Summer Research Fellowship Programme

Certificate

This is to certify that Mr Rajat Kumar worked on a project entitled "Ordinary differential equations" in online mode during June – July 2021 as a Summer Research Fellow under the supervision of Dr Jagmohan Tyagi, Indian Institute of Technology, Gandhinagar. The Summer Research Fellowship Programme is jointly sponsored by IASc (Bengaluru), INSA (New Delhi) and NASI (Prayagraj).

Place: Bengaluru
Date: 27-08-2021

P. K Das
Chairman, Science Education Panel

INDIAN ACADEMY OF SCIENCES, C. V. RAMAN AVENUE, POST BOX No. 8005,
RAMAN RESEARCH INSTITUTE CAMPUS, SADASHIVANAGAR P.O.,
BENGALURU 560 080, INDIA

EDVISION LABS

presents

Certificate of Completion

to

KRITI VERMA

for successfully completing her tenure as **Marketing & Research Consultant** and **Graphic Designer** from 1st July, 2021 to 12th September, 2021.

SAMRIDH NANGIA

FOUNDER

GOLD

Certificate

OF INTERNSHIP

THIS IS TO CERTIFY THAT

Aakanksha

FROM FARIDABAD WAS ASSOCIATED
WITH AASHMAN FOUNDATION IN THE CAPACITY OF AN INTERN
FROM MAY 2021 TO OCTOBER 2021 WITH OUR
HR MANAGER GROUP.

HE/SHE HAS COMPLETED HIS/HER INTERNSHIP
WITH A THREE STAR PERFORMANCE

OCTOBER 21

aashman FOUNDATION

Munishpundir
FOUNDER / DIRECTOR

Ref No. : SP77/DMI-F/MD/00021 **Date:** 2 July 2021

Experience Letter

Congratulations!

Dear Priya Saini,

This letter is to certify that **Ms. Priya Saini, Digital Marketing Intern** has Successfully completed her project with Super77. Her Project tenure was from **1 April 2021 to 30 June 2021**. She was the part of **Marketing Department** and was actively involved in the projects assigned to Her.

During the span, we found Her punctual and hardworking person. Her learning powers are good and she picks up swifty. Her feedback and evaluation proved that she learned keenly.

We wish Her a bright future.

Mentor:- Rachit Goyal, Marketing Department

For Human Charging India Pvt. Ltd

Amit Pandey
CEO, Super77

A Mentorship Program
SUPER77
BY HUMAN CHARGER
Transforming Individuals to Succeed

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Haryana, 122002, India
CIN No.:- U80904HR2020PTC088999

Email:- Hello@super77.in
Website:- Super77.in

Summer Research Fellowship Programme

Certificate

This is to certify that Mr Prassanna Nand Jha worked on a project entitled "On the distribution of primes" in online mode during May - July 2021 as a Summer Research Fellow under the supervision of Dr Brundaban Sahu, National Institute of Science Education & Research, Bhubaneswar. The Summer Research Fellowship Programme is jointly sponsored by IASc (Bengaluru), INSA (New Delhi) and NASI (Prayagraj).

Place: Bengaluru
Date: 24-08-2021

P. K. Das
Chairman, Science Education Panel

INDIAN ACADEMY OF SCIENCES, C. V. RAMAN AVENUE, POST BOX No. 8005,
RAMAN RESEARCH INSTITUTE CAMPUS, SADASHIVANAGAR P.O.,
BENGALURU 560 080, INDIA

Indian Academy of Sciences, Bengaluru
Indian National Science Academy, New Delhi
The National Academy of Sciences India, Prayagraj

SUMMER RESEARCH FELLOWSHIPS — 2021

Format for the Final-week Report*

Name of the candidate : PRASSANNA NAND JHA

Application Registration no. : MATS109 - I

Date of Commencement of work : 31/05/2021

Mode of work : From Home: ✓ Guide's Laboratory: ☐

Date of completion : 27/07/2021

Total no. of days worked : 58 DAYS

Name of the guide : DR. BRUNDABAN SAHU

Guide's institution : NISER, BHUBANESWAR

Project title : ON THE DISTRIBUTION OF PRIMES

Address with pin code to which the certificate could be sent:

SARASWATI SADAN, HARIHAR BARI, B.DEOGHAR - 814112

E-mail ID: prassannanandjha.1@gmail.com

Phone No: 9934392734

TA Form attached with final report (not applicable for those **working from home**) : YES ______ NO ✓

If, NO, Please specify reason : WORKING FROM HOME

Prassanna Nand Jha
Signature of the candidate
Date: 27/07/2021

Signature of the guide
Date: 28/07/2021

*The final report could be anywhere between 20 and 25 pages including tables, figures etc. This format should be the first page of the report and should be stapled with the main report.

(For office use only; do not fill/tear)

Candidate's name:	Fellowship amount:
Student: Teacher:	Deduction:
Guide's name:	TA fare:
KVPY Fellow: INSPIRE Fellow:	Amount to be paid:
PFMS Unique Code:	A/c holder's name:
Others	

Summer Research Fellowship Programme 2021

On the Distribution of Primes

(Final Report)

Student:
Prassanna Nand Jha
Department of Mathematics
Sri Venkateswara College, Delhi

Mentor:
Dr. Brundaban Sahu
Department of Mathematics
NISER, Bhubaneswar

Date of submission: 28th July 2021

Abstract

We study the analytic proof of the prime number theorem given by Jacques Hadamard and Charles Poussin in the year 1896. We begin by reformulating the problem using complex analysis and introduce the Riemann Zeta function $\zeta(s)$. We then establish some of its useful properties and study the behaviour of $\zeta(s)$ near the line $\sigma = 1$. Finally we show that $\zeta(s)$ has no zeros of the form $1 + it$ and conclude the proof.

Keywords: Prime number theorem, Riemann zeta function, analytic number theory, distribution of primes

Background

The prime number theorem first appeared in 1798 as a conjecture by the French mathematician Legendre. On the basis of his study of a table of primes up to $1,000,000$, Legendre stated that if x is not greater than $1,000,000$, then $\frac{x}{\log(x)-1.08366}$ is very close to the number of primes less than x. The German mathematician Gauss also conjectured an equivalent of this theorem in his notebook, perhaps prior to 1800.

Theorem: Let $\pi(x)$ be the prime-counting function that gives the number of primes less than or equal to x, for any real number x. Then, $\frac{x}{\log(x)}$ is a good approximation to $\pi(x)$, i.e.

$$\lim_{x\to\infty} \frac{\pi(x)}{\left[\frac{x}{\log(x)}\right]} = 1 \text{ or } \pi(x) \sim \frac{x}{\log(x)}$$

In his two papers from 1848 and 1850, the Russian mathematician Chebyshev was able to prove unconditionally that the ratio $\frac{\pi(x)}{x/\log(x)}$ is bounded above and below by two explicitly given constants near 1, for all sufficiently large x.

Then, in 1859, the mathematician Riemann introduced new ideas into the subject, chiefly that the distribution of prime numbers is intimately connected with the zeros of the analytically extended Riemann zeta function of a complex variable. Extending Riemann's ideas, two proofs of the asymptotic law of the distribution of prime numbers were found independently by Jacques Hadamard and Charles Jean de la Vallée Poussin and appeared in the same year (1896).

1 Reduction of the problem

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we define the following functions introduced by Chebyshev:

$$\psi(x) = \sum_{n \le x} \Lambda(n) \text{ where } \Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^\alpha; \alpha \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

$$\theta(x) = \sum_{p \le x} \log(p) \text{ where } p \text{ is a prime.}$$

Lemma 1.1. *For $x \in \mathbb{R}$,*

$$\psi(x) = \sum_{1 \,\le\, \alpha \,\le\, \log_2 x} \theta(x^{1/\alpha})$$

Proof. By the definition of $\psi(x)$, it can be rewritten as

$$\psi(x) = \sum_{n \le x \ ; \ n = p^\alpha} \log(p) = \sum_{\alpha=1}^{\infty} \sum_{p \le x^{1/\alpha}} \log(p)$$

But $2 \le p$ and we must have $p \le x^{1/\alpha}$. So must restrict α so that $2 \le x^{1/\alpha}$. Hence

$$\psi(x) = \sum_{1 \,\le\, \alpha \,\le\, \log_2 x} \ \sum_{p \le x^{1/\alpha}} \log(p) = \sum_{1 \,\le\, \alpha \,\le\, \log_2 x} \theta(x^{1/\alpha})$$

□

Theorem 1.1. $\psi(x) \sim x \iff \theta(x) \sim x$

Proof. By Lemma 1.1, we have :

$$\begin{aligned} \psi(x) - \theta(x) &= \sum_{2 \,\le\, \alpha \,\le\, \log_2 x} \theta(x^{1/\alpha}) \\ &\le \sum_{2 \,\le\, \alpha \,\le\, \log_2 x} x^{1/\alpha} \log(x^{1/\alpha}) \\ &\le \sum_{2 \,\le\, \alpha \,\le\, \log_2 x} x^{1/2} \log(x^{1/2}) \\ &= \frac{\sqrt{x} \log^2(x)}{2 \log(2)} \end{aligned}$$

Hence, $\frac{\psi(x)}{x} - \frac{\theta(x)}{x} \le \frac{\log^2(x)}{2\sqrt{x}\log(2)}$. So, by squeeze theorem, we get

$$\lim_{x \to \infty} \frac{\psi(x)}{x} - \frac{\theta(x)}{x} = 0, \text{ which implies } \psi(x) \sim x \iff \theta(x) \sim x$$

□

Lemma 1.2. *For $x \geq 2$, $\theta(x) = O(x)$*

Proof. First of all, we observe that for $n \geq 2$ the product of all primes between n and $2n$ is a factor of $\binom{2n}{n}$. Hence, as $\binom{2n}{n} \leq 2^{2n}$, we obtain:

$$\prod_{n\,<\,p\,\leq\,2n} p \;\leq\; 2^{2n} \implies \sum_{n\,<\,p\,\leq\,2n} \log(p) \;\leq\; 2n\log(2)$$

So we get the following list of inequalities:

$$\begin{aligned}
\theta(2n) - \theta(n) &\leq 2n\log(2) \\
\theta(n) - \theta\left(\frac{n}{2}\right) &\leq n\log(2) \\
\theta\left(\frac{n}{2}\right) - \theta\left(\frac{n}{4}\right) &\leq \frac{n}{2}\log(2) \\
&\vdots
\end{aligned}$$

Adding all of them, we obtain : $\theta(2n) \leq [2n + n + \frac{n}{2} + \cdots]\log(2)$.
So, $\theta(2n) \leq 4n\log(2)$ or $\theta(n) \leq 2\log(2)n$. Thus, $\theta(x) = O(x)$. □

Theorem 1.2. $\theta(x) \sim x \iff \pi(x) \sim \frac{x}{\log(x)}$

Proof. We define two sequence a_n and b_n such that

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases} \implies \sum_{n\leq x} a_n = \pi(x)$$

$$b_n = \log(n)a_n \implies \sum_{n\leq x} b_n = \theta(x)$$

Then, if we take $f(t) = \frac{1}{\log(t)}$, we will have (by Abel's summation[1]):

$$\sum_{n\leq x} b(n)f(n) = B(x)f(x) - \int_2^x B(t)f'(t)dt, \text{ where } B(x) = \sum_{n\leq x} b_n$$

On substituting, we get:

$$\pi(x) = \frac{\theta(x)}{\log(x)} + \int_2^x \frac{\theta(t)}{t\log^2 t}dt$$

$$\implies \frac{\pi(x)\log(x)}{x} = \frac{\theta(x)}{x} + \frac{\log(x)}{x}\int_2^x \frac{\theta(t)}{t\log^2 t}dt$$

[1]For any arithmetical function $a(n)$ let $A(x) = \sum_{n\leq x} a(n)$ where $A(x) = 0$ if $x < 1$. If f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$, then we have

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt$$

Now, by Lemma 1.2, we have the following result:

$$\begin{aligned}
\int_2^x \frac{\theta(t)}{t\log^2 t}dt &= O\left(\int_2^x \frac{t}{t\log^2 t}dt\right) \\
&= O\left(\int_2^{\sqrt{x}} \frac{1}{\log^2 2}dt + \int_{\sqrt{x}}^x \frac{1}{\log^2 \sqrt{x}}dt\right) \\
&= O\left(\frac{\sqrt{x}}{\log^2 2} + \frac{x-\sqrt{x}}{\log^2 \sqrt{x}}\right)
\end{aligned}$$

So:

$$\frac{\log(x)}{x}\int_2^x \frac{\theta(t)}{t\log^2 t}dt = O\left(\frac{\log(x)}{\sqrt{x}\log^2 2} + \frac{4\left[1-\frac{1}{\sqrt{x}}\right]}{\log(x)}\right)$$

Hence, we observe:

$$\lim_{x\to\infty}\frac{\pi(x)\log(x)}{x} = \lim_{x\to\infty}\frac{\theta(x)}{x} \text{ , i.e. } \theta(x)\sim x \iff \pi(x)\sim\frac{x}{\log(x)}$$

$\square$

Thus, we have established this beautiful three-way asymptotic equivalence :

$$\psi(x)\sim x \iff \theta(x)\sim x \iff \pi(x)\sim\frac{x}{\log(x)}$$

Lemma 1.3. *Let a_n be a nonnegative sequence and define:*

$$A(x) = \sum_{n\le x} a_n \text{ and } A_1(x) = \int_1^x A(t)dt$$

Then,

$$\sum_{n\le x}(x-n)a_n = A_1(x)$$

and if $A_1(x)\sim Lx^c$ as $x\to\infty$ for some $c>0$ and $L>0$, then $A(x)\sim cLx^{c-1}$ as $x\to\infty$.

Proof. We know that by Abel summation, for any function $f(n)$,

$$\sum_{n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt$$

Taking $f(n)=n$, we get:

$$\sum_{n\le x} na_n = xA(x) - \int_1^x A(t)dt \implies \sum_{n\le x}(x-n)a_n = A_1(x)$$

Now, we choose arbitrary constants $0 < \alpha < 1$ and $\beta > 1$, and observe:

$$\begin{aligned} A_1(\beta x) - A_1(x) &= \int_x^{\beta x} A(t)dt \geq (\beta x - x)A(x) \\ \implies \frac{1}{\beta - 1}\left\{\frac{A_1(\beta x)}{x^c} - \frac{A_1(x)}{x^c}\right\} &\geq \frac{A(x)}{x^{c-1}} \end{aligned}$$

and

$$\begin{aligned} A_1(x) - A_1(\alpha x) &= \int_{\alpha x}^{x} A(t)dt \leq (x - \alpha x)A(x) \\ \implies \frac{1}{1 - \alpha}\left\{\frac{A_1(x)}{x^c} - \frac{A_1(\alpha x)}{x^c}\right\} &\leq \frac{A(x)}{x^{c-1}} \end{aligned}$$

So, as $x \to \infty$, we obtain:

$$\begin{aligned} \limsup_{x\to\infty} \frac{A(x)}{x^{c-1}} &= L\,\frac{\beta^c - 1}{\beta - 1} \\ \liminf_{x\to\infty} \frac{A(x)}{x^{c-1}} &= L\,\frac{1 - \alpha^c}{1 - \alpha} \end{aligned}$$

Now, if $\alpha \to 1-$ and $\beta \to 1+$, then,

$$\begin{aligned} \limsup_{x\to\infty} \frac{A(x)}{x^{c-1}} &= L\,\lim_{\beta\to 1+} \frac{\beta^c - 1^c}{\beta - 1} = L\left[\frac{d(\beta^c)}{d\beta}\right]_{\beta=1} = Lc \\ \liminf_{x\to\infty} \frac{A(x)}{x^{c-1}} &= L\,\lim_{\alpha\to 1-} \frac{1^c - \alpha^c}{1 - \alpha} = L\left[\frac{d(\alpha^c)}{d\alpha}\right]_{\alpha=1} = Lc \end{aligned}$$

Hence,

$$\lim_{x\to\infty} \frac{A(x)}{x^{c-1}} = Lc \text{ or } A_1(x) \sim Lx^c \implies A(x) \sim cLx^{c-1}$$

. ◻

Theorem 1.3. *Define a smoothing function*

$$\psi_1(x) = \int_1^x \psi(t)dt.$$

Then, $\psi_1(x) \sim \frac{x^2}{2} \implies \psi(x) \sim x$

Proof. Let $a_n = \Lambda(n)$ in Lemma 1.3. Then we get $\psi_1(x) \sim \frac{x^2}{2} \implies \psi(x) \sim x$ ◻

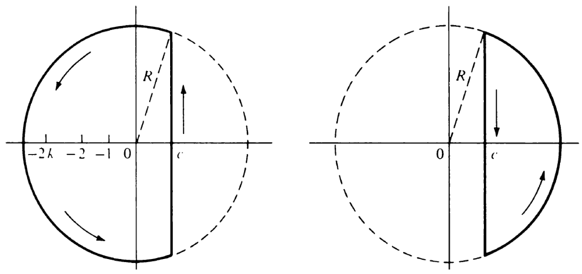
2 Representation as a contour integral

Lemma 2.1. *Let $z \in \mathbb{C}$. If $c > 0$ and $u > 0$, then for every $k \in \mathbb{N}$, we have:*

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)\cdots(z+k)}dz = \begin{cases} \frac{1}{k!}(1-u)^k & \text{if } 0 < u \le 1 \\ 0 & \text{if } u > 1 \end{cases},$$

the integral being absolutely convergent.

Proof. We first define two different contours for $0 < u \le 1$ and $u > 1$ as follows. Here, $R > 2k$.



$0 < u \le 1$ $\qquad\qquad$ $u > 1$

Now, if $|z| = R$, then for integers $1 \le n \le k$, we shall have
$|z+n| \ge |z| - n = R - n \ge R - k \ge R - \frac{R}{2} \implies \frac{1}{|z+n|} \le \frac{2}{R}$.

Let $z = x + iy$. We observe that given our contours, if $0 < u \le 1$, then $x \le c$ and if $u > 1$, then $x \ge c$. So, in both cases, $u^{-x} \le u^{-c}$. Hence :

$$\begin{aligned} \left|\frac{u^{-z}}{z(z+1)\cdots(z+k)}\right| &\le \frac{|u^{-x}|}{|z||z+1|\cdots|z+k|} \qquad \because |u^{-iy}| = 1 \\ &\le \frac{2^k u^{-c}}{R^{k+1}} \end{aligned}$$

Thus, if we integrate this expression over any circular arc, the integral will be dominated by the expression $2\pi R\ \frac{2^k u^{-c}}{R^{k+1}} = O(R^{-k})$ and this tends to 0 as $R \to \infty$. So, if C_R is any of the two contours shown in the figure above, we shall have :

$$\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)\cdots(z+k)}dz = \int_{C_R} \frac{u^{-z}}{z(z+1)\cdots(z+k)}dz \text{ as } R \to \infty$$

Now we observe that the integrand has simple poles at integers $0, -1, -2, \cdots, -k$ and all these points lie outside the contour for $u > 1$ and inside the contour for $0 < u \le 1$. Hence, if $u > 1$, Cauchy's integral theorem yields :

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)\cdots(z+k)}dz = 0$$

But if we have $0 < u \leq 1$, then Cauchy's residue theorem yields:

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)\cdots(z+k)}dz = \sum_{n=0}^{k} \mathrm{Res}\left(\frac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)}, -n\right)$$

Now, we recall that if a function F has a simple pole at z_0 and G is analytic at z_0, then $\mathrm{Res}(F \cdot G, z_0) = G(z_0) \cdot \mathrm{Res}(F, z_0)$. So,

$$\begin{aligned}\mathrm{Res}\left(\frac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)}, -n\right) &= \left[\frac{u^{-z}}{\Gamma(z+k+1)}\right]_{z=-n} \cdot \mathrm{Res}(\Gamma(z), -n) \\ &= \frac{u^{n}}{\Gamma(k+1-n)} \cdot \frac{(-1)^n}{n!}\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)\cdots(z+k)}dz &= \sum_{n=0}^{k} \frac{(-u)^n}{(k-n)!n!} \\ &= \frac{1}{k!}\sum_{n=0}^{k}\binom{k}{n}(-u)^n \\ &= \frac{1}{k!}(1-u)^k\end{aligned}$$

Hence proved:
If $c > 0$ and $u > 0$, then for every $k \in \mathbb{N}$, we have:

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)\cdots(z+k)}dz = \begin{cases} \frac{1}{k!}(1-u)^k & \text{if } 0 < u \leq 1 \\ 0 & \text{if } u > 1 \end{cases},$$

the integral being absolutely convergent. ∎

Lemma 2.2. *For $s \in \mathbb{C}$ with $Re(s) > 1$, the Riemann Zeta function is defined as:*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Then,

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} \ \textit{where } \Lambda(n) = \begin{cases} \log(p) & \textit{if } n = p^{\alpha}; \alpha \in \mathbb{N} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We have

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

$$\implies \left(1-\frac{1}{2^s}\right)\zeta(s) = 1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\cdots$$
$$\implies \left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s) = 1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\cdots$$
$$\implies \cdots\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s) = 1$$

This holds true because we know that $\sum\left|\frac{1}{p^s}\right|$ converges uniformly for $\sigma > 1$ and hence the infinite product $\prod\left(1-\frac{1}{p^s}\right)$ also converges absolutely. Besides, the product does not converge to zero and hence we can write

$$\begin{aligned}\frac{1}{\zeta(s)} &= \prod_{p\ prime}\left(1-\frac{1}{p^s}\right)\\ &= 1+\sum_{n\ prime}\left(-\frac{1}{n^s}\right)+\sum_{n\ =\ p_1p_2}\left(-\frac{1}{p_1^s}\cdot-\frac{1}{p_2^s}\right)+\cdots\\ &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\end{aligned}$$

where $\mu(n) = \begin{cases} 1 & \text{if } n=1 \\ (-1)^k & \text{if } a_1=a_2=\cdots=a_k=1 \quad \text{for } n=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k} \\ 0 & \text{otherwise}\end{cases}$

Now Weierstrass's Theorem for Series states that if the functions $f_1(z), f_2(z), f_3(z), \cdots$ are holomorphic in an open set D and $F(z) = \sum_{i=1}^{\infty} f_i(z)$ converges uniformly on every closed and bounded subset of D, then $F(z)$ is holomorphic on D and for all $k \geq 1$, the series $\sum_{i=1}^{\infty} f_i^{(k)}(z)$ converges on every closed and bounded subset of D with limit $F^{(k)}(z)$. So, as $\frac{1}{n^s}$ is holomorphic if $\sigma > 1\ \forall\ n \in \mathbb{N}$ and the function $\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges uniformly for $\sigma > 1$, we can write:

$$\zeta'(s) = \sum_{n=1}^{\infty}\frac{d}{ds}\frac{1}{n^s} = -\sum_{n=1}^{\infty}\frac{\log(n)}{n^s}.$$

Now we know that for two sequences $a(n)$ and $b(n)$,

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s}\sum_{n=1}^{\infty}\frac{b(n)}{n^s} = \sum_{n=1}^{\infty}\left\{\frac{\sum_{d|n}a(d)b(\frac{n}{d})}{n^s}\right\}.$$

Hence,

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty}\left\{\frac{\sum_{d|n}\mu(d)\log(\frac{n}{d})}{n^s}\right\} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.$$

$\square$

Theorem 2.1. *If $c > 1$ and $x \geq 1$, we have:*

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

Proof. By Lemma 1.3, we know that :

$$\psi_1(x) = \sum_{n\leq x}(x-n)\cdot\Lambda(n)$$

$$\implies \frac{\psi_1(x)}{x} = \sum_{n\leq x}\left(1-\frac{n}{x}\right)\cdot\Lambda(n)$$

And, if we take $u = \frac{n}{x}$ and $k = 1$, then Lemma 2.1 gives us :

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{(x/n)^s}{s(s+1)}ds = \begin{cases} (1-\frac{n}{x}) & \text{if } n \leq x \\ 0 & \text{if } n > x \end{cases}, \text{ for } c > 0$$

So,

$$\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{\Lambda(n)\cdot(x/n)^s}{s(s+1)}ds\right] = \sum_{n\leq x}\left(1-\frac{n}{x}\right)\cdot\Lambda(n) = \frac{\psi_1(x)}{x}$$

Now, we note that the partial sums satisfy:

$$\begin{aligned}\sum_{k=1}^{N}\left|\int_{c-\infty i}^{c+\infty i}\frac{\Lambda(k)\cdot(x/k)^s}{s(s+1)}ds\right| &= \sum_{k=1}^{N}\int_{c-\infty i}^{c+\infty i}\frac{\Lambda(k)\cdot(x/k)^c}{|s||s+1|}ds \\ &= \sum_{k=1}^{N}\frac{\Lambda(k)}{k^c}\int_{c-\infty i}^{c+\infty i}\frac{x^c}{|s||s+1|}ds \leq A\sum_{k=1}^{N}\frac{\Lambda(k)}{k^c}\end{aligned}$$

where A is a constant. So, the term

$$\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{\Lambda(n)\cdot(x/n)^s}{s(s+1)}ds\right]$$

converges absolutely and hence the summation and integral can be interchnaged. Hence,

$$\frac{\psi_1(x)}{x} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^s}{s(s+1)}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}ds \text{ for } c > 0$$

Finally, by Lemma 2.2, we get:

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds \text{ for } c > 1$$

□

3 Removal of the pole

Lemma 3.1. *If a non-zero function $f(s)$ has a pole of order k at $s = a$ then the quotient $\frac{f'(s)}{f(s)}$ has a first order pole at $s = a$ with residue $-k$.*

Proof. As $f(s)$ has a pole of order k at $s = a$, we can write $f(s) = \frac{g(s)}{(s-a)^k}$ where g is analytic at a. Using the quotient rule, we get

$$\begin{aligned} f'(s) &= \frac{(s-a)^k g'(s) - k(s-a)^{k-1} g(s)}{(s-1)^{2k}} \\ &= \frac{g'(s)}{(s-a)^k} - \frac{kg(s)}{(s-a)^{k+1}} \\ &= \frac{g(s)}{(s-1)^k}\left[\frac{g'(s)}{g(s)} - \frac{k}{s-a}\right] \end{aligned}$$

Therefore, $\frac{f'(s)}{f(s)} = \frac{g'(s)}{g(s)} - \frac{k}{s-a}$. Since $g(s) \neq 0$ and is analytic, it follows that $\frac{g'(s)}{g(s)}$ is analytic at $s = a$. Hence, $\frac{f'(s)}{f(s)}$ has a first order pole at $s = a$ and the residue is given by :

$$\text{Res}\left[\frac{f'(s)}{f(s)}, s = a\right] = \lim_{s \to a}(s-a)\frac{f'(s)}{f(s)} = -k$$

. ∎

Theorem 3.1. *$\frac{\zeta'(s)}{\zeta(s)}$ has a first order pole at $s = 1$ with residue -1 and hence $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ is analytic at $s = 1$.*

Proof. We define the Dirichlet Eta function as :

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

for a complex number s with $Re(s) > 0$ and note that $\eta(1)$ converges to $\log(2)$ by the Maclaurian series expansion of $\log(1 + x)$. Now,

$$\begin{aligned} \zeta(s) - \eta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right) - \left(1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots\right) \\ &= 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right) \\ &= \frac{1}{2^{s-1}}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right) \\ &= \frac{1}{2^{s-1}}\zeta(s) \end{aligned}$$

Thus,

$$\zeta(s) = \frac{\eta(s)}{1 - \frac{1}{2^{s-1}}}$$

and by this expression, $\zeta(s)$ has a first pole at $s = 1$. To find the residue, we compute:

$$\begin{aligned}
\lim_{s\to 1}(s-1)\zeta(s) &= \lim_{s\to 1} \frac{(s-1)\eta(s)}{1 - \frac{1}{2^{s-1}}} \\
&= \eta(1) \lim_{s\to 1} \frac{2^{s-1}(s-1)}{2^{s-1} - 1} \\
&= \log(2) \lim_{s\to 0} \frac{2^s(s)}{2^s - 1} \\
&= \log(2) \cdot \frac{1}{\log(2)} \\
&= 1.
\end{aligned}$$

Therefore $\zeta(s)$ has a residue of 1 at $s = 1$ and by the definition of the Riemann zeta function, $\zeta(s) \neq 0$ for $Re(s) > 1$. Hence Lemma 3.1 implies that $\frac{\zeta'(s)}{\zeta(s)}$ has a first order pole at $s = 1$ with residue -1. So, $\frac{\zeta'(s)}{\zeta(s)} - \frac{-1}{s-1}$ is analytic at $s = 1$. Thus,

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \text{ is analytic at } s = 1.$$

□

In Theorem 2.1, we proved that

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds \text{ for } c > 1.$$

Let $g(s) = \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)$ and $s = c + it$. Then, it can be rewritten as :

$$\frac{\psi_1(x)}{x^2} = \frac{x^{c-1}}{2\pi} \int_{-\infty}^{+\infty} g(c+it) e^{it\log(x)} dt \text{ for } c > 1$$

We want to prove $\frac{\psi_1(x)}{x^2} \sim \frac{1}{2}$ and so we let $x \to \infty$. Now the Riemann-Lebesgue lemma in the theory of Fourier series states that

$$\lim_{x\to\infty} \int_{-\infty}^{+\infty} f(t) e^{itx} dt = 0$$

if the integral $\int_{-\infty}^{+\infty} |f(t)| dt$ converges.

But by Theorem 3.1, the term $g(c+it)$ has a pole at $s=1$, and so $\int_{-\infty}^{+\infty} |g(c+it)|dt$ doesn't converge for $c > 1$. Thus, we will first subtract the pole at $s=1$ from $\frac{\zeta'(s)}{\zeta(s)}$.

We get the following theorem:

Theorem 3.2. *If $c > 1$ and $x \geq 1$ we have*

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right) ds$$

Proof. Let $c > 0$ and $x \geq 1$. By Lemma 2.1 for $k=2$ and $u = 1/x$, we have:

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)(s+2)} ds = \frac{1}{2}\left(1 - \frac{1}{x}\right)^2$$

Repplacing s by $s-1$ in the integral,

$$\frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{(s-1)(s)(s+1)} ds \text{ for } c > 1$$

We already had

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds \text{ for } c > 1.$$

So, for $c > 1$ and $x \geq 1$,

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right) ds$$

□

Now we let $h(s) = \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)$ and $s = c+it$. Then, it can be rewritten as :

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{x^{c-1}}{2\pi}\int_{-\infty}^{+\infty} h(c+it)e^{it\log(x)}dt \text{ for } c > 1$$

If we let $x \to \infty$, and show that $\int_{-\infty}^{+\infty} |h(c+it)|dt$ converges then by Riemann Lebesgue Lemma, $\int_{-\infty}^{+\infty} h(c+it)e^{it\log(x)}dt \to 0$. But the term $\frac{x^{c-1}}{2\pi} \to \infty$ we get the indeterminate form $\infty \cdot 0$.

To deal with this, we show that it is possible to replace c by 1 in

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right) ds$$

and that the integral $\int_{-\infty}^{+\infty} |h(1+it)|dt$ converges.

This requires a detailed study of the Riemann Zeta function in the vicinity of the line $\sigma = 1$ for $s = \sigma + it$.

4 Riemann Zeta Function near $\sigma = 1$

Lemma 4.1. *Let $s = \sigma + it \in \mathbb{C}$. For any $N \in \mathbb{N}$ and $\sigma > 0$ we have:*

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - s \int_N^\infty \frac{x - [x]}{x^{s+1}} dx + \frac{N^{1-s}}{s-1}$$

Proof. Euler's summation formula states that if f is smooth on $[a, b]$, $0 < a < b$, then

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t - [t])f'(t)dt + f(x)([x] - x) - f(y)([y] - y)$$

Taking $f(n) = n^{-s}$, $y = N$ and $x \to \infty$, we get for $\sigma > 1$:

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \sum_{n>N} \frac{1}{n^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \int_N^\infty \frac{1}{t^s} dt + \int_N^\infty \frac{-s(t-[t])}{t^{s+1}} dt + \lim_{x\to\infty} \frac{[x] - x}{x^s} - \frac{[N] - N}{N^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \left[\frac{t^{1-s}}{1-s}\right]_N^\infty - s \int_N^\infty \frac{(t-[t])}{t^{s+1}} dt + \lim_{x\to\infty} \frac{[x] - x}{x^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^\infty \frac{(t-[t])}{t^{s+1}} dt
\end{aligned}$$

So, we have proved that for $\sigma > 1$,

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - s \int_N^\infty \frac{x - [x]}{x^{s+1}} dx + \frac{N^{1-s}}{s-1}$$

If $0 < \sigma \leq 1$, $\exists$ a $\delta > 0$ such that $\sigma \geq \delta$ and hence

$$\int_N^\infty \frac{(t-[t])}{t^{s+1}} dt \leq \int_N^\infty \frac{1}{t^{\delta+1}} dt.$$

So, the integral $\int_N^\infty \frac{(t-[t])}{t^{s+1}} dt$ converges uniformly for $\sigma \geq \delta$ and hence represents an analytic function in the half plane $\sigma > 0$. Therefore, by analytic continuation, for $\sigma > 0$,

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - s \int_N^\infty \frac{x - [x]}{x^{s+1}} dx + \frac{N^{1-s}}{s-1}.$$

$\square$

Theorem 4.1. *For every $A > 0$, there exists a constant M such that $|\zeta(s)| \leq M\log(t)$ with $\sigma \geq \frac{1}{2}$ satisfying $\sigma > 1 - \frac{A}{\log(t)}$ and $t \geq e$.*

Proof. If $\sigma \geq 2$, we have

$$|\zeta(s)| = \left|\sum_{n=1}^{\infty} \frac{1}{n^s}\right| = \sum_{n=1}^{\infty} \frac{1}{|n^{\sigma+it}|} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$$

and so $|\zeta(s)| \leq M\log(t)$ holds trivially.

Therefore we assume $\sigma < 2$ and $t \geq e$. We then have $|s| = |\sigma+it| \leq |\sigma|+|t| < 2+t < 2t$ and $|s-1| \geq |(\sigma-1)+it| = \sqrt{(\sigma-1)^2+t^2} \geq t$. By Lemma 4.1, for any $N \in \mathbb{N}$ and $\sigma > 0$ we have:

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - s\int_N^\infty \frac{x-[x]}{x^{s+1}}dx + \frac{N^{1-s}}{s-1}.$$

Then

$$\begin{aligned}
|\zeta(s)| &\leq \left|\sum_{n=1}^{N} \frac{1}{n^s}\right| + \left|s\int_N^\infty \frac{x-[x]}{x^{s+1}}dx\right| + \left|\frac{N^{1-s}}{s-1}\right| \\
&\leq \sum_{n=1}^{N} \frac{1}{n^\sigma} + 2t\int_N^\infty \frac{1}{x^{\sigma+1}}dx + \frac{N^{1-\sigma}}{t} \\
&= \sum_{n=1}^{N} \frac{1}{n^\sigma} + \frac{2t}{\sigma N^\sigma} + \frac{N^{1-\sigma}}{t}
\end{aligned}$$

We take $N = [t]$ and so $\log(n) \leq \log(t)$ if $n \leq N$. Besides, we also choose $\sigma \geq \frac{1}{2}$ satisfying $\sigma > 1 - \frac{A}{\log(t)}$ and $t \geq e$. So,

$$\frac{1}{n^\sigma} = \frac{n^{1-\sigma}}{n} < \frac{1}{n}e^{A\frac{\log(n)}{\log(t)}} \leq \frac{1}{n}e^A = O\left(\frac{1}{n}\right).$$

Now as $N \leq t < N+1$ and $\sigma \geq \frac{1}{2}$, we get

$$\frac{2t}{\sigma N^\sigma} \leq \frac{2(N+1)}{\frac{1}{2}N} = O\left(1+\frac{1}{N}\right) = O(1) \text{ and } \frac{N^{1-\sigma}}{t} = \frac{N}{t}\frac{1}{N^\sigma} = O\left(\frac{1}{N}\right) = O(1).$$

Hence

$$|\zeta(s)| \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} + O(1) = O\left(\sum_{n=1}^{N} \frac{1}{n}\right) + O(1) = O(\log(N)) + O(1) = O(\log(t)).$$

So, this implies that for every $A > 0$, there exists a constant M such that $|\zeta(s)| \leq M\log(t)$ with $\sigma \geq \frac{1}{2}$ satisfying $\sigma > 1 - \frac{A}{\log(t)}$ and $t \geq e$.

□

Theorem 4.2. *For every $A > 0$, there exists a constant M such that $|\zeta'(s)| \leq M\log^2(t)$ with $\sigma \geq \frac{1}{2}$ satisfying $\sigma > 1 - \frac{A}{\log(t)}$ and $t \geq e$.*

Proof. If $\sigma \geq 2$, we have

$$|\zeta'(s)| = \left|\sum_{n=1}^{\infty} -\frac{\log(n)}{n^s}\right| = \sum_{n=1}^{\infty} \frac{\log(n)}{|n^{\sigma+it}|} = \sum_{n=1}^{\infty} \frac{\log(n)}{n^{\sigma}} \leq \sum_{n=1}^{\infty} \frac{\log(n)}{n^2} = |\zeta'(2)|$$

and so $|\zeta'(s)| \leq M\log^2(t)$ holds trivially. Therefore we assume $\sigma < 2$ and $t \geq e$ and this gives us the inequalities $|s| < 2t$ and $|s-1| \geq t$.

We had shown that for any $N \in \mathbb{N}$ and $\sigma > 0$,

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - s\int_N^{\infty} \frac{x-[x]}{x^{s+1}}dx + \frac{N^{1-s}}{s-1}.$$

This yields:

$$\begin{aligned}\zeta'(s) = &-\sum_{n=1}^{N} \frac{\log(n)}{n^s} + s\int_N^{\infty} \frac{(x-[x])\log(x)}{x^{s+1}}dx \\ &- \int_N^{\infty} \frac{x-[x]}{x^{s+1}}dx - \frac{N^{1-s}\log(N)}{s-1} - \frac{N^{1-s}}{(s-1)^2}.\end{aligned}$$

So,

$$\begin{aligned}|\zeta'(s)| &\leq \left|\sum_{n=1}^{N} \frac{\log(n)}{n^s}\right| + \left|s\int_N^{\infty} \frac{(x-[x])\log(x)}{x^{s+1}}dx\right| + \left|\int_N^{\infty} \frac{x-[x]}{x^{s+1}}dx\right| \\ &\quad + \left|\frac{N^{1-s}\log(N)}{s-1}\right| + \left|\frac{N^{1-s}}{(s-1)^2}\right| \\ &\leq \log(N)\sum_{n=1}^{N} \frac{1}{n^{\sigma}} + 2t\int_N^{\infty} \frac{\log(x)}{x^{\sigma+1}}dx + \int_N^{\infty} \frac{1}{x^{\sigma+1}}dx + \frac{N^{1-\sigma}\log(N)}{t} + \frac{N^{1-\sigma}}{t^2} \\ &= \log(N)\sum_{n=1}^{N} \frac{1}{n^{\sigma}} + 2t\left(\frac{\log(N)}{\sigma N^{\sigma}} + \frac{1}{\sigma^2 N^{\sigma}}\right) + \frac{1}{\sigma N^{\sigma}} + \frac{N^{1-\sigma}\log(N)}{t} + \frac{N^{1-\sigma}}{t^2}\end{aligned}$$

We take $N = [t]$ and so $\log(n) \leq \log(t)$ if $n \leq N$. Besides, we also choose $\sigma \geq \frac{1}{2}$ satisfying $\sigma > 1 - \frac{A}{\log(t)}$ and $t \geq e$. So, we get the relation $\frac{1}{n^{\sigma}} = O\left(\frac{1}{n}\right)$. Thus

$$
\begin{aligned}
|\zeta'(s)| &= O\left(\log(N)\sum_{n=1}^{N}\frac{1}{n}\right) + O\left(\frac{t\log(N)}{\sigma N}\right) + O\left(\frac{t}{\sigma^2 N}\right) \\
&+ O\left(\frac{1}{\sigma N}\right) + O\left(\frac{N^{1-\sigma}\log(N)}{t}\right) + O\left(\frac{N^{1-\sigma}}{t^2}\right) \\
&= O\left(\log^2(N)\right) + O\left(\log(N)\right) + O(1) \\
&= O\left(\log^2(t)\right)
\end{aligned}
$$

So. this implies that for every $A > 0$, there exists a constant M such that $|\zeta'(s)| \leq M\log^2(t)$ with $\sigma \geq \frac{1}{2}$ satisfying $\sigma > 1 - \frac{A}{\log(t)}$ and $t \geq e$.

□

Lemma 4.2. *If $\sigma > 1$ we have $\zeta^3(\sigma)\ |\zeta(\sigma+it)|^4\ |\zeta(\sigma+2it)| \geq 1$.*

Proof. By Lemma 2.2, we have that if $\sigma > 1$,

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.$$

Hence for $\sigma > 1$

$$
\begin{aligned}
\log(\zeta(s)) &= \int \frac{\zeta'(s)}{\zeta(s)}ds \\
&= \int\left(-\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\right)ds \\
&= -\sum_{n=2}^{\infty}\int\frac{\Lambda(n)}{n^s}ds \qquad \because \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s} \text{ converges uniformly for } \sigma > 0 \\
&= \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\log(n)} + C \qquad \text{where } C \text{ is a constant.}
\end{aligned}
$$

So, we get that $\zeta(s) = e^{G(s)}$ where $G(s) = \displaystyle\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\log(n)} + C$ for some constant C.

Thus, if $\sigma \to \infty$, then $G(s) \to C$ or $\zeta(s) \to e^C$. But we know that $\zeta(s) \to 1$ as $\sigma \to \infty$. This implies $e^C = 1$ or $C = 0$.

So, we have established that $\zeta(s) = e^{G(s)}$ where

$$G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log(n)} = \sum_p \sum_{m=1}^{\infty} \frac{\log(p)}{(p^m)^s \log(p^m)} = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{m\sigma} e^{imt\log(p)}}.$$

This implies

$$|\zeta(s)| = \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{e^{-imt\log(p)}}{mp^{m\sigma}}\right) = \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{\cos(mt\log(p))}{mp^{m\sigma}}\right).$$

We apply this formula repeatedly for $s = \sigma$, $s = \sigma + it$ and $s = \sigma + 2it$, and obtain

$$\begin{aligned}
&\zeta^3(\sigma)\ \ |\zeta(\sigma+it)|^4\ \ |\zeta(\sigma+2it)| \\
&= \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{3}{mp^{m\sigma}}\right) \cdot \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{4\cos(mt\log(p))}{mp^{m\sigma}}\right) \cdot \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{\cos(2mt\log(p))}{mp^{m\sigma}}\right) \\
&= \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{3 + 4\cos(mt\log(p)) + \cos(2mt\log(p))}{mp^{m\sigma}}\right)
\end{aligned}$$

But we have $3+4\cos(\theta)+\cos(2\theta) = 3+4\cos(\theta)+2\cos^2(\theta)-1 = 2(1+\cos(\theta)^2 \geq 0$. Therefore each term in the infinite series is nonnegative and hence

$$\zeta^3(\sigma)\ \ |\zeta(\sigma+it)|^4\ \ |\zeta(\sigma+2it)| \geq 1.$$

□

Theorem 4.3. $\zeta(1+it) \neq 0\ \ \forall\ \ t \in \mathbb{R}$

Proof. If $t = 0$, the relation is trivial and therefore we assiume $t \neq 0$. Lemma 4.2 yields that if $\sigma > 1$, then $\zeta^3(\sigma)\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)| \geq 1$ or

$$\{(\sigma-1)\zeta(\sigma)\}^3 \left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 |\ |\zeta(\sigma+2it)| \geq \frac{1}{\sigma-1}.$$

Now if we let $\sigma \to 1+$, we will have $(\sigma-1)\zeta(\sigma) \to 1$ since $\zeta(s)$ has residue 1 at the pole $s = 1$. Besides $\zeta(\sigma+2it) \to \zeta(1+2it)$. Now if we assume that for some $t_0 \neq 0$, we have $\zeta(1+it_0) = 0$, then

$$\lim_{\sigma\to 1+} \left|\frac{\zeta(\sigma+it_0)}{\sigma-1}\right|^4 = \lim_{\sigma\to 1+} \left|\frac{\zeta(\sigma+it_0) - \zeta(1+it_0)}{\sigma-1}\right|^4 = |\zeta'(1+it_0)|^4.$$

Hence we obtain $|\zeta'(1+it_0)|^4|\zeta(1+2it_0)| \geq \lim_{\sigma\to 1+} \frac{1}{\sigma-1}$, which tends to $+\infty$. So we arrive a contradiction, i.e. our assumption was wrong. Thus proved: $\zeta(1+it) \neq 0\ \ \forall\ \ t \in \mathbb{R}$.

□

Theorem 4.4. *There is a constant $M > 0$ such that $\left|\frac{1}{\zeta(s)}\right| < M\log^7(t)$ and $\left|\frac{\zeta'(s)}{\zeta(s)}\right| < M\log^9(t)$ whenever $\sigma \geq 1$ and $t \geq e$.*

Proof. For $\sigma \geq 2$ we have $\left|\frac{1}{\zeta(s)}\right| = \left|\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right| \leq \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2)$ and so $\left|\frac{1}{\zeta(s)}\right| < M\log^7(t)$ trivially. Let $1 < \sigma \leq 2$ and $t \geq e$. By Lemma 4.2, we have

$$\zeta^3(\sigma)\ |\zeta(\sigma+it)|^4\ |\zeta(\sigma+2it)| \geq 1 \text{ or } \frac{1}{|\zeta(\sigma+it)|} \leq \zeta^{3/4}(\sigma)|\zeta(\sigma+2it)|^{1/4}$$

Now as $(\sigma-1)\zeta(\sigma)$ is bounded in the interval $1 \leq \sigma \leq 2$, there exists an absolute constant M such that $(\sigma-1)\zeta(\sigma) \leq M$ and hence $\zeta(\sigma) \leq \frac{M}{\sigma-1}$.

Also, by Theorem 4.1, $|\zeta(\sigma+2it)| = O(\log(t))$ if $1 \leq \sigma \leq 2$. Hence, there exists an absolute constant A such that $\frac{1}{|\zeta(\sigma+it)|} \leq \frac{A\log^{1/4}(t)}{(\sigma-1)^{3/4}}$.
Therefore, for some constant $B > 0$ we have

$$|\zeta(\sigma+it)| > \frac{B(\sigma-1)^{3/4}}{\log^{1/4}(t)}, \text{ if } 1 < \sigma \leq 2 \text{ and } t \geq e.$$

We also note that this inequality holds trivially when $\sigma = 1$. Hence, we have established that for $1 \leq \sigma \leq 2$ and $t \geq e$, we have $|\zeta(\sigma+it)| > \frac{B(\sigma-1)^{3/4}}{\log^{1/4}(t)}$.

Now, assume that $1 \leq \sigma \leq 2$ and $t \geq e$ and let α be any real satisfying $1 < \alpha < 2$. Then either $1 \leq \sigma \leq \alpha < 2$ or $1 < \alpha \leq \sigma \leq 2$.

If $1 \leq \sigma \leq \alpha < 2$, $t \geq e$, then Theorem 1.8 yields that for some constant $K_1 > 0$,

$$|\zeta(\alpha+it) - \zeta(\sigma+it)| \leq \int_{\sigma}^{\alpha} |\zeta'(u+it)|du \leq (\alpha-1)K_1\log^2(t).$$

Hence, by triangle inequality,

$$\begin{aligned}
|\zeta(\sigma+it)| &\geq |\zeta(\alpha+it)| - |\zeta(\sigma+it) - \zeta(\alpha+it)| \\
&\geq |\zeta(\alpha+it)| - (\alpha-1)K_1\log^2(t) \\
&> \frac{B(\alpha-1)^{3/4}}{\log^{1/4}(t)} - (\alpha-1)K_1\log^2(t)
\end{aligned}$$

And if $1 < \alpha \leq \sigma \leq 2$, then $t \geq e$, then

$$|\zeta(\sigma+it)| > \frac{B(\sigma-1)^{3/4}}{\log^{1/4}(t)} \geq \frac{B(\alpha-1)^{3/4}}{\log^{1/4}(t)} \geq \frac{B(\alpha-1)^{3/4}}{\log^{1/4}(t)} - (\alpha-1)K_1\log^2(t).$$

Combining both we conclude that if we have $1 \leq \sigma \leq 2$ and $t \geq e$, then for every $\alpha \in (1,2)$, there exists some constant $K > 0$ such that

$$|\zeta(\sigma+it)| \geq \frac{B(\alpha-1)^{3/4}}{\log^{1/4}(t)} - (\alpha-1)K\log^2(t)$$

If $B < 2K$, let $t_0 = e^{\left(\frac{B}{2K}\right)^{4/9}}$. Then we choose $\alpha = 1 + \left(\frac{B}{2K}\right)^4 \frac{1}{\log^9(t)}$ and notice that $1 < \alpha < 2$ for all $t > t_0$. And since $t_0 < e$, we will have $1 < \alpha < 2$ for all $t \geq e$. So we can substitute this value of α in the inequality to obtain:

$$\begin{aligned}
|\zeta(\sigma+it)| &\geq \frac{B\left(\left(\frac{B}{2K}\right)^4 \frac{1}{\log^9(t)}\right)^{3/4}}{\log^{1/4}(t)} - \left(\frac{B}{2K}\right)^4 \frac{1}{\log^9(t)} K\log^2(t) \\
&\geq \frac{B^4}{8K^3}\frac{1}{\log^7(t)} - \frac{B^4}{16K^3}\frac{1}{\log^7(t)} \\
&= \frac{B^4}{16K^3}\frac{1}{\log^7(t)} = \frac{C}{\log^7(t)} \text{ where } C > 0
\end{aligned}$$

And if $B \geq 2K$, we choose $\alpha = 1 + \frac{1}{2\log^9(t)}$ to have $1 < \alpha < 2$ for all $t \geq e$. Upon substitution we get:

$$\begin{aligned}
|\zeta(\sigma+it)| &\geq \frac{B\left(\frac{1}{2\log^9(t)}\right)^{3/4}}{\log^{1/4}(t)} - \frac{1}{2\log^9(t)}K\log^2(t) \\
&\geq \frac{B}{2^{3/4}}\frac{1}{\log^7(t)} - \frac{K}{2}\frac{1}{\log^7(t)} \\
&= \frac{1}{2^{3/4}}\left(B - \frac{K}{2^{1/4}}\right)\frac{1}{\log^7(t)} = \frac{C'}{\log^7(t)} \text{ where } C' > 0
\end{aligned}$$

So, we have established that there exists a constant $M > 0$ such that $\left|\frac{1}{\zeta(s)}\right| < M\log^7(t)$ whenever $1 \leq \sigma \leq 2$ and $t \geq e$. Previously we had showed that this also holds for $\sigma \geq 2$. Hence we can say that there exists a constant $M > 0$ such that $\left|\frac{1}{\zeta(s)}\right| < M\log^7(t)$ whenever $\sigma > 1$ and $t \geq e$.

Now by Theorem 4.2, there exists another constant N such that $|\zeta'(s)| \leq N\log^2(t)$ whenever $\sigma \geq 1$ and $t \geq e$. So, we can have a positive constant $P = MN$ such that $\frac{\zeta'(s)}{\zeta(s)} < P\log^9(t)$.

Thus proved: $\left|\frac{1}{\zeta(s)}\right| = O(\log^7(t))$ and $\left|\frac{\zeta'(s)}{\zeta(s)}\right| = O(\log^9(t))$ whenever $\sigma \geq 1$ and $t \geq e$.

$\square$

5 Revisiting the contour integral

Theorem 5.1. *For $x \geq 1$ we have*

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(1+it)e^{it\log(x)}dt$$

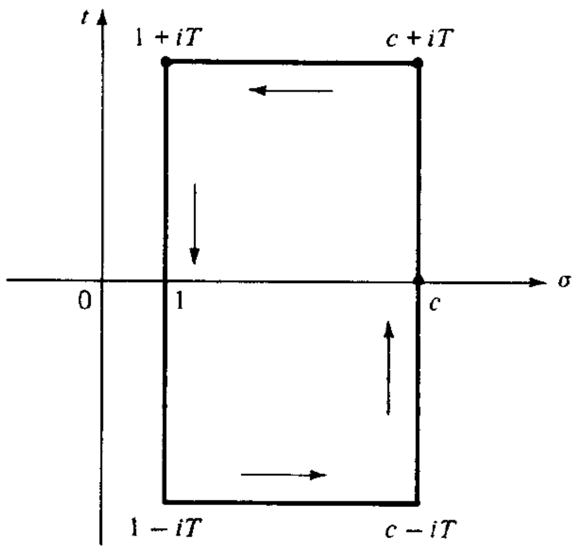
where the integral $\int_{-\infty}^{+\infty} |h(1+it)|dt$ converges.

Proof. In Theorem 3.2, we had proved that if $c > 1$ and $x \geq 1$ we have

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^{s-1}h(s)ds$$

where $h(s) = \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)$

We now consider the following rectangular contour R on the complex plane.



Now as the term $x^{s-1}h(s)$ is analytic inside and on R, Cauchy's theorem gives $\int_R x^{s-1}h(s) = 0$. So,

$$\int_1^c x^{\sigma-1-iT}h(\sigma-iT)d\sigma + \int_{-T}^{+T} x^{c-1+it}h(c+it)dt + \int_c^1 x^{\sigma-1+iT}h(\sigma+iT)d\sigma + \int_{+T}^{-T} x^{it}h(1+it)dt = 0$$

or

$$\int_1^c x^{\sigma-1-iT}h(\sigma-iT)d\sigma + \int_{-T}^{+T} x^{c-1+it}h(c+it)dt = \int_1^c x^{\sigma-1+iT}h(\sigma+iT)d\sigma + \int_{-T}^{+T} x^{it}h(1+it)dt$$

Now we note that for $s = \sigma + iT$ and for $s = \sigma - iT$,

$$|h(s)| \leq \left|\frac{1}{s(s+1)}\right| \cdot \left|\frac{\zeta'(s)}{\zeta(s)}\right| + \left|\frac{1}{(s-1)s(s+1)}\right| \leq \frac{1}{T^2}\left|\frac{\zeta'(s)}{\zeta(s)}\right| + \frac{1}{T^3}$$

And so by Theorem 4.4, if $T \geq e$, $|h(s)| = O\left(\frac{\log^9(T)}{T^2}\right) + O\left(\frac{1}{T^3}\right) = O\left(\frac{\log^9(T)}{T^2}\right)$. Hence

$$\int_1^c x^{\sigma-1-iT} h(\sigma - iT) d\sigma = O\left(\int_1^c x^{c-1} \frac{M \log^9(T)}{T^2}\right) = O\left((c-1)x^{c-1}\frac{M\log^9(T)}{T^2}\right)$$

and

$$\int_1^c x^{\sigma-1+iT} h(\sigma + iT) d\sigma = O\left(\int_1^c x^{c-1} \frac{M \log^9(T)}{T^2}\right) = O\left((c-1)x^{c-1}\frac{M\log^9(T)}{T^2}\right).$$

Therefore, as $T \to \infty$ these integrals tend to 0. Hence we can write

$$\int_{-\infty}^{+\infty} x^{c-1+it} h(c+it) dt = \int_{-\infty}^{+\infty} x^{it} h(1+it) dt.$$

And therefore,

$$\begin{aligned}
\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 &= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^{s-1} h(s) ds \\
&= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} x^{it} h(1+it) dt \\
&= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{it\log(x)} h(1+it) dt
\end{aligned}$$

Besides, $|h(1+it)| = O\left(\frac{M\log^9(t)}{t^2}\right)$ and as

$$\int_{-\infty}^{+\infty} |h(1+it)| dt = \int_{-\infty}^{-e} |h(1+it)| dt + \int_{-e}^{+e} |h(1+it)| dt + \int_{+e}^{+\infty} |h(1+it)| dt,$$

the integral $\int_{-\infty}^{+\infty} |h(1+it)| dt$ converges.

□

Theorem 5.2. *For $x \geq 1$, $\psi_1(x) \sim \frac{x^2}{2}$. This implies that $\pi(x) \sim \frac{x}{\log(x)}$ and hence the Prime Number Theorem is proved.*

Proof. In Theorem 5.1, we proved for $x \geq 1$ we have

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(1+it)e^{it\log(x)}dt$$

where the integral $\int_{-\infty}^{+\infty} |h(1+it)|dt$ converges.

Now by Riemann Lebesgue Lemma, $\int_{-\infty}^{+\infty} h(1+it)e^{it\log(x)}dt$ converges to 0 as $x \to 0$. Hence we get:

$$\lim_{x\to\infty}\left(\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2\right) = 0 \text{ or } \lim_{x\to\infty}\frac{\psi_1(x)}{x^2} = \frac{1}{2}.$$

This implies: $\psi_1(x) \sim \frac{x^2}{2}$. Now by Theorem 1.1, 1.2 and 1.3 we can say that

$$\psi_1(x) \sim \frac{x^2}{2} \implies \psi(x) \sim x \implies \phi(x) \sim x \implies \pi(x) \sim \frac{x}{\log(x)}.$$

Therefore the Prime Number Theorem is proved. $\square$

Conclusion

As elegant as this proof is, the method relies highly on complex analysis although the statement of this theorem does not itself involve complex numbers. So in search of a simpler proof, mathematicians Atle Selberg and Erdős published new, independent elementary proofs of the prime number theorem in 1948 using properties of logarithms. These proofs enticed other mathematicians to consider similar methods for number theory conjectures previously considered too profound for such seemingly simple methods. Many exciting results followed, including Helmut Maier's 1985 elementary proof showing unexpected irregularities in the distribution of primes. Then in 1980, Newman gave a shorter proof that provided a much simpler link between the zeta function and the prime number theorem. So to sum up, the Prime Number Theorem is one of the most interesting gems of number theory and mathematicians may never stop searching for new and more illuminating paths to the prime number theorem.

Acknowledgements

I would wish to extend my gratitude to my mentor Prof. Brundaban Sahu, for his constant encouragement, co-operation and invaluable guidance throughout the time. I am also thankful to the Indian Academy of Sciences, Indian National Science Academy and The National Academy of Sciences, India for giving me an opportunity to learn and grow as a young researcher. Besides, I would like to thank my friends and family members for being patient, supportive and helping me out in every possible way.

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CERTIFICATE OF **APPRECIATION**

Presented to **Akanksha Singh**

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Date: 19/06/2021

TO WHOM IT MAY CONCERN

This is to certify that Ms. Akanksha Singh, D/O Mr. Rakesh Kumar Singh, a student of Sri Venkateswara College Delhi University , has successfully completed 02 (Two) month (from 15th April, 2021 to 15th June, 2021) long Internship programme at SAKSHAM FOUNDATION. During the period of her internship programme with us she was Found punctual, hardworking and inquisitive.

We wish her every success in life.

SAKSHAM FOUNDATIO

Authorized Signature

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Date: 19/06/2021

CERTIFICATE

This is to certify that **Ms. Akansha Singh D/o Mr. Rakesh Kumar Singh** has been an active volunteer in our organization since April, 2021. She has actively volunteered in our campaign "**Mission Pahel**". During her internship she performed many activities like preparing fund raising posters, case studies of children and also raised Rs. 3,250/- which helps Saksham Foundation to provide dry ration kits to nearly 07 families. She taught underprivileged children for 2 months during this period. She has taught 40 children's in this period. This not only helps them but has also provided Kavya with invaluable experience. The main motive was to make them understand easy tricks of Maths etc. to underprivileged children who do not even have access to basic facilities like food, water and sanitation. With the help of volunteers like Akansha our campaign has been successful. She actively taught these children with full passion and zeal during this whole campaign.

During this campaign, she always had a smile on her face and a desire to help the underprivileged section. Her passion of teaching she was able to make a small impact in our society which may give fruitful results to our country in future. She is a dedicated girl and has great potential in her. During this campaign she suggested various unconventional techniques and ideas to teach them and these turned out to be successful.

She has indeed played a major role in giving their precious time to community service. Akansha has been instrumental in bringing new hopes in the lives of these underprivileged children. Our organization is proud to have a volunteer like Akansha who played a vital role in making this campaign a success.

I am proud to say that Akansha is one of the few who have given themselves sincerely to the massive tasks ahead of all of us. This is indeed commendable and I can only say that she is worthy of the highest recommendation and has set an example for others to follow.

SAKSHAM FOUNDATION Regd. No. 1465/4 New Delhi

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GC

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RECOMMENDATION LETTER

To Whom It May Concern

My name is **Ravinder Pathania** and I highly recommend **Ridam Kapoor** who has worked under me as an aspiring content writter intern over a span of one month.

I can assure you that **Ridam Kapoor** has good work ethics and exemplary skills. She is a constant top performer and she has exceeded in her work.

She is resourceful, effective and a solution oriented person. Her learnability has always been a remarkable quality.

I strongly recommend **Ridam Kapoor** as an excellent and professional worker. I am willing to provide more information if needed.

Warm Regards

Ravinder Pathania
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MBD Group
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LETTER OF COMPLETION

To Whom it May Concern

We are glad to inform you that **Ms. Ridam Kapoor** from Sri Venkateswara College (New Delhi) has successfully completed her internship at 'MBD Group' from 20th December, 2021 – 24th January, 2022.

During her internship, she was exposed to research and paper work on Mathematics book for 8th standard.

We found her extremely inquisitive and hard working. She was very much interested to learn the functions of our core division and also willing to put her best efforts and get in depth of the subject to understand it better.

Her association with us was very fruitful and we wish her all the best in her future endeavours.

Ravinder Pathania
(GM Publication)
MBD Group of India
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The Polymath Jr program - jayati

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Inbox 7,021

Starred

Snoozed

Important

Chats

Sent

Meet

New meeting

Join a meeting

Hangouts

Jayati

Arav Agarwal
lol

Katha Rajan
You were on a video call

17 of 28

The Polymath Jr program

Inbox ×

Polly Matthews <thepolymathjr@gmail.com> Sat, 10 Apr 2021, 07:19

to adam.sheffer, zberikkyzy, Steven, Johanna.N.Franklin, cdoneill, Patrick, brubaker, Alexandra, Yunus, Jonathan.Farley, vhm, sk206, zehrabasiyanik, abd

Dear Polymath Jr applicants,

If you are receiving this email, then **you are officially accepted to the Polymath JR program.** We are excited to do some serious math work with you during the summer!

We received more applications than we expected. To make sure that the participants have a good experience, we don't want the research groups to be too large. For that reason, many other people are currently on a waiting list. We are recruiting more professors, who would run more projects, which would hopefully allow us to accept more applicants.

If you intend not to participate in the program, please let us know as soon as possible. For example, if you plan to participate in another summer program. That would allow us to accept people from the waitlist.

Applicants who are no longer students in the fall of 2021 or who didn't start college yet were automatically placed on the waiting list. We might have missed a few of those. **If you are no longer an undergrad in the fall of 2021 or have not started college yet, please let us know now.** Contact Adam at Adam.Sheffer@baruch.cuny.edu.

The program will run from June 21st to August 15th. We are now recruiting more staff and building other parts of the program. So you probably won't hear from us for a while. Possibly for over a month. If you have any questions, you are welcome to write to Adam.

At this point, no one is assigned a project. The first week of the program is dedicated to learning about the projects and ranking your preferences. That means that you cannot prepare towards your project yet. If you do want to prepare, we recommend practicing your proof writing skills. This would be important for almost all projects.

Looking forward to meeting you!

The Polymath Jr team

Welcome to the Computational Algebra project!   Inbox

Alexandra Seceleanu <aseceleanu@unl.edu>   Sat, 26 Jun 2021, 10:36

to aryamanmaithani.iitb@gmail.com, shivam.j.mohite@Vanderbilt.Edu, manavbatavia@gmail.com, audric.lebovitz2002@gmail.com, WANGSHES@UCI.E

Hello everyone,

If you are receiving this email, you have been assigned to Polymath's Computational Algebra team. Welcome aboard! There are 20 students on this team currently.

The first order of business is to set up meeting times for the next weeks of the program. To help us do this, please fill out the poll at the following link (type your name on the left and click sign in) https://www.when2meet.com/?12193319-LKR8A

I am planning two meetings for next week: a tutorial in Macaulay2 and a general meeting to discuss the exercises in the introductory document. Now that you know you have been assigned to this project, please start working on these exercises. You are encouraged to collaborate and share your solutions in the following Overleaf document https://www.overleaf.com/project/60b511a08bb39205efb83610 (click on Exercises.tex and then Recompile)

As always, if you have questions please reach out on Discord or via email.

Alexandra

Indian Academy of Sciences, Bengaluru
Indian National Science Academy, New Delhi
The National Academy of Sciences India, Prayagraj

SUMMER RESEARCH FELLOWSHIPS — 2021

Format for the Final-week Report*

Name of the candidate	:	JAYATI SOOD
Application Registration no.	:	MATS489
Date of Commencement of work	:	01/06/2021
Mode of work	:	From Home: ✓ Guide's Laboratory:
Date of completion	:	27/07/2021
Total no. of days worked	:	56
Name of the guide	:	DR ARVIND AYYER
Guide's institution	:	INDIAN INSTITUTE OF SCIENCE, BENGALURU
Project title	:	ALTERNATING SIGN MATRICES

Address with pin code to which the certificate could be sent:

3, SOUTH PARK APARTMENTS, KALKAJI
NEW DELHI - 110019

E-mail ID: jayatisood9@gmail.com

Phone No: 9582696780

TA Form attached with final report (not applicable for those **working from home**) : YES ______ NO ✓

If, NO, Please specify reason : WORKING FROM HOME

Jyoti

Signature of the candidate

Date: 06/11/2021

A S Ayyer

Signature of the guide

Date: 9/11/2021

*The final report could be anywhere between 20 and 25 pages including tables, figures etc

This format should be the first page of the report and should be stapled with the main report.

(For office use only: do not fill/tear)

Candidate's name:	Fellowship amount:
Student: Teacher:	Deduction:
Guide's name:	TA fare:
KVPY Fellow: INSPIRE Fellow:	Amount to be paid:
PFMS Unique Code:	A/c holder's name:
Others	

Alternating Sign Matrices

Student: Jayati Sood Mentor: Dr. Arvind Ayyer

ABSTRACT

An Alternating Sign Matrix (ASM) is a square matrix consisting of 0's, 1's and -1's such that the entries in each row and each column sum to 1 and the nonzero entries in each row and each column alternate in sign. They arise naturally in the evaluation of λ-determinants, which are a generalisation of determinants obtained by modifying the Dodgson algorithm for determinant evaluation. Just as regular determinants may be expressed as a sum over permutation matrices, these λ-determinants can be expressed as a sum over alternating sign matrices.

These generalizations of permutation matrices have connections with various combinatorical objects. This report details the topics covered over the course of SRFP 2021, in a guided study of concepts in algebraic combinatorics, from the perspective of alternating sign matrices.

Contents

1. Introduction

Definition 1.1. An alternating sign matrix is a square matrix consisting of 0's, 1's and -1's such that the entries in each row and each column sum to 1 and the nonzero entries in each row and each column alternate in sign.

Example 1.2.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1.1. **The ASM Conjecture.**

Let A_n the number of $n \times n$ alternating sign matrices, and let $A_{n,k}$ denote the number of $n \times n$ ASMs such that the $(1,k)^{\text{th}}$ entry is 1. D.Robbins, H.Rumsey, and W. Mills discovered that dividing the set of $n \times n$ ASMs into classes according to the position of 1 in the first row results in a Pascal's triangle-like pattern:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 3 & & 2 & & & \\
 & & 7 & & 14 & & 14 & & 7 & & \\
 & 42 & & 105 & & 135 & & 105 & & 42 & \\
429 & & 1287 & & 2002 & & 2002 & & 1287 & & 429
\end{array}$$

Figure 1. Number of ASMs of order n such that $a_{1,k} = 1$

$$\begin{array}{ccccccccc}
 & & & & 2/2 & & & & \\
 & & & 2/3 & & 3/2 & & & \\
 & & 2/4 & & 5/5 & & 4/2 & & \\
 & 2/5 & & 7/9 & & 9/7 & & 5/2 & \\
2/6 & & 9/14 & & 16/16 & & 14/9 & & 6/2
\end{array}$$

Figure 2. Ratios of adjacent terms from Figure 1

They also found that ratios of horizontally adjacent entries, themselves form a pattern (Fig 2). The n^{th} row starts with $2/(n+1)$ and ends with $(n+1)/2$. The striking observation is that each ratio appears to arise from the two ratios diagonally above, by adding numerators and adding denominators. This led to the following conjecture:

Conjecture 1.3. ***(The refined ASM conjecture)*** *For $1 \leqslant k < n$,*

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{\binom{n-2}{k-1} + \binom{n-1}{k-1}}{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}$$

Mills, Robbins and Rumsey also conjectured the formula for computing A_n. This was proven by Doron Zeilberger ('96), and separately by Greg Kuperberg ('96).

Theorem 1.4. ***(The ASM theorem)*** *The total number of $n \times n$ alternating sign matrices is*

$$A_n = A_{n+1,1} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

1.2. **Objects enumerated by ASMs.**

1.2.1. *Descending plane partitions.* A descending plane partition (DPP) of order n is a 2-dimensional array of positive integers less than or equal to n such that the left-hand edges are successively indented, there is weak decrease across rows and strict decrease down columns, and the number of entries in each row is strictly less than the largest entry in that row. George Andrews in 1979 conjectured a formula for counting DPPs of order n, which when computed for small values of n, was found to equal the the number of ASMs of the same order.

1.2.2. *Totally symmetric self-complementary plane partitions.* A totally symmetric self- complementary plane partition (TSSCPP) is a plane partition that is symmetric, cyclically symmetric, and equal to its complement. Andrews proved in 1994, that the number of TSSCPPs in a $2n \times 2n \times 2n$ room is equal to the number of ASMs of order n.

1.2.3. *Alternating sign triangles.* An alternating sign triangle (AST) of order n is a triangular array $(a_{i,j})_{1\leqslant i\leqslant n,\ i-n\leqslant j\leqslant n-i}$ with entries in $\{0, \pm 1\}$, such that all (i) row sums are 1, (ii) non-zero entries alternate in each row and column, and (iii) the topmost non-zero entry in each column is 1, if it exists. In 2020, A. Ayyer, R. E. Behrend, and I. Fischer showed that the number of ASTs of order n is equal to the number of ASMs of the same order.

While A_n enumerates these objects, a bijection between any pair of these has not been found.

1.2.4. *Aztec diamonds.* For an ASM A of order n, let $\mu(A)$ be the number of (-1) entries in A. Then, a 2-enumeration of ASMs is $\sum 2^{\mu(A)} = 2^{n(n+1)/2}$. This is also the number of domino tilings possible for an Aztec diamond if order n.

2. Dodgson Condensation (1866)

Dodgson condensation is an algorithmic technique to evaluate the determinant of an $n \times n$ matrix by iteratively computing 2×2 determinants. The algorithm itself is an application of the previously established Desnanot-Jacobi adjoint matrix theorem (1833). The definition of the determinant may be generalised by modifying Dodgson's algorithm in a specific way to obtain λ-determinants (as defined further), the evaluation of which first gave rise to alternating sign matrices.

Let $M = (m_{i,j})$ be an $n \times n$ matrix. For $S, T \in [n] \equiv \{1, \ldots, n\}$, let M_T^S be the matrix obtained by deleting the rows in S and columns in T from M. If $S = \{i\}$ and $T = \{j\}$, then M_i^j is the matrix that remains when the i^{th} row and j^{th} column of M are deleted.

Definition 2.1. The *cofactor matrix* M^C of an $n \times n$ matrix M is defined as follows:

$$(M^C)_{i,j} = (-1)^{i+j}|M_i^j|$$

Multiplying M by its cofactor matrix M^C, we get for $i = j$

$$\begin{aligned}(M \cdot M^C)_{i,i} = \sum_{k=1}^{n} M_{i,k}(M^C)_{k,i} &= (-1)^{i+1}(m_{i,1}|M_1^i| - m_{i,2}|M_2^i| + \ldots + (-1)^{n-1}m_{i,n}|M_n^i| \\ &= |M|\end{aligned}$$

And for $i \neq j$ we get the determinant of the matrix with row i replaced with a duplicate of row j, and hence

$$\begin{aligned}(M \cdot M^C)_{i,j} = \sum_{k=1}^{n} M_{j,k}(M^C)_{k,i} &= (-1)^{j+1}(m_{j,1}|M_1^i| - m_{j,2}|M_2^i| + \ldots + (-1)^{n-1}m_{j,n}|M_n^i| \\ &= 0\end{aligned}$$

$$\therefore M \cdot M^C = \begin{pmatrix} |M| & 0 & 0 & \cdots & 0 \\ 0 & |M| & 0 & \cdots & 0 \\ 0 & 0 & |M| & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & |M| \end{pmatrix}$$

$$\begin{aligned}
&\implies \quad |M| \cdot |M^C| = |M|^n \\
&\implies \quad |M^C| = |M|^{n-1}
\end{aligned}$$

Considered in the polynomial ring $\mathbb{C}[m_{1,1}, m_{1,2}, \ldots, m_{n,n}]$ where all "m"s are treated as formal variables.

Theorem 2.2. ***(Desnanot-Jacobi adjoint matrix theorem)*** *If M is an $n \times n$ matrix, then*

$$|M||M_{1,n}^{1,n}| = |M_1^1||M_n^n| - |M_n^1||M_1^n|$$

Proof. Starting with M^C, replace the $(i,j)^{\text{th}}$ entry with $\delta_{i,j}$ (where δ is the Kronecker delta) to obtain a new matrix M^*, and evaluate its determinant:

$$|M^*| = \begin{vmatrix}
|M_1^1| & 0 & 0 & \cdots & 0 & (-1)^{n+1}|M_1^n| \\
-|M_2^1| & 1 & 0 & \cdots & 0 & (-1)^{n+2}|M_2^n| \\
|M_3^1| & 0 & 0 & \cdots & 0 & (-1)^{n+3}|M_3^n| \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^n|M_{n-2}^1| & 0 & 0 & \cdots & 1 & -|M_{n-1}^n| \\
(-1)^{n+1}|M_n^1| & 0 & 0 & \cdots & 0 & -|M_n^n|
\end{vmatrix}$$

$= |M_1^1||M_n^n| - |M_n^1||M_1^n|$ Multiplying M and M^* and taking the determinant, we get

$$|M \cdot M^*| = \begin{vmatrix}
|M| & m_{1,2} & m_{1,3} & \cdots & 0 \\
0 & m_{2,2} & m_{2,3} & \cdots & 0 \\
0 & m_{3,2} & m_{3,3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & m_{n,2} & m_{n,3} & \cdots & |M|
\end{vmatrix}$$

$= |M|^2|M_{\{1,n\}}^{\{1,n\}}|$

Equating the determinants, we have shown

$$|M||M_{\{1,n\}}^{\{1,n\}}| = |M_1^1||M_n^n| - |M_n^1||M_1^n|$$

□

Dodgson's Algorithm

Dodgson realised that the Desnanot-Jacobi theorem could be expressed in the form of an algorithm to compute the determinant of an $n \times n$ matrix:

$$|M| = \frac{|M_1^1||M_n^n| - |M_n^1||M_1^n|}{|M_{\{1,n\}}^{\{1,n\}}|}$$

This definition of the determinant can be generalised as follows.

Definition 2.3. The λ-determinant of an $n \times n$ matrix is given by

$$|M|_\lambda = \frac{|M_1^1|_\lambda|M_n^n|_\lambda + \lambda|M_n^1|_\lambda|M_1^n|_\lambda}{|M_{\{1,n\}}^{\{1,n\}}|_\lambda}$$

with $|\phi|_\lambda = \lambda,$ and $|a|_\lambda = a$.

Remark 2.4. $|a_{i,j}|$ is a Laurent polynomial of λ for any n.

Theorem 2.5. ***(Robbins-Rumsey, '86):*** *Let M be an $n \times n$ matrix with entries $a_{i,j}$, $\mathcal{A}_n$ the set of $n \times n$ alternating sign matrices, $\mathcal{I}(B)$ the inversion number of B, and $N(B)$ the number of -1s in B. Then,*

$$|M|_\lambda = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)}(1+\lambda^{-1})^{N(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}}$$

The alternating sign matrix conjecture was thus motivated by counting the number of summands in the above expression.

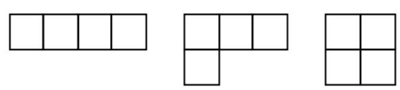
3. Plane Partitions

George Andrews in 1979 conjectured a formula for counting descending plane partitions (ref). It was observed that the number of descending plane partitions of order n, was the same as the number of ASMs of the same order. While numerical evidence suggests a bijection between ASMs and DPPs, attempts to establish such a bijection have so far been unsuccessful.

To understand plane partitions, we first define partitions:

Definition 3.1. A *partition* λ of a non-negative integer n is a sequence $(\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k$ satisfying $\lambda_1 \leqslant \ldots \leqslant \lambda_k$ and $\sum \lambda_i = n$

Partitions can be represented by their Young diagrams, where each part λ_i is represented by the i^{th} row of λ_i unit squares. Each row is left-justified:



4 3+1 2+2

Figure 3. Young diagrams of some partitions of 4

Definition 3.2. A *plane partition* is a rectangular array of non-negative integers $\pi = \pi(i,j)_{i,j \geqslant 1}$ satisfying $\pi_{i,j} \geqslant \pi_{i,j1}$, and $\pi_{i,j} \geqslant \pi_{i+1,j}$.

Equivalently, a plane partition is an arrangement of unit cubes in a room, stacked against one corner.

3.1. Generating function for plane partitions.

MacMahon showed that the generating function for plane partitions can be expressed as

$$\sum_{n=1}^{\infty} pp(n)q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}$$

Let $pp(n)$ =number of plane partitions of n, and let $|\pi| := \sum_{i,j \geqslant 1} \pi_{i,j} = n$.

Definition 3.3. A *standard Young tableau (SYT)* of a partition $\lambda \vdash n$ is a filling of its Young diagram with unique entries in $[n]$ such that the entries increase along each row from left to right, and along each column from top to bottom.

Definition 3.4. The *shape* of an SYT is the partition λ, the Young diagram of which the SYT is a filling of.

Definition 3.5. For a cell c in the Young diagram of λ, let the *hook* of c be

$$H(c) = \{\text{cells to the right of } c \text{ in its row and to the bottom of } c \text{ in its column}\}$$

Let

$$\begin{aligned} f^\lambda &= \text{No. of SYTs of shape } \lambda, \\ h(c) &= |H(c)| \end{aligned}$$

then,

Theorem 3.6. ***(Hook-length formula, Frame-Robinson-Thrall, 1959):*** *For any partition $\lambda \vdash n$ and c a cell in the Young Diagram of λ,*

$$f^\lambda = \frac{n!}{\prod\limits_c h(c)}$$

3.1.1. *The Robinson-Schensted-Knuth (RSK) Algorithm.*
There exists a bijection between pairs of SYTs (P, Q) of the same shape $\lambda \vdash n$ and permutations $\pi \in S_n$. This can be proven using the *Robinson-Schensted-Knuth (RSK) algorithm.*

Definition 3.7. Define a *near Young tableau (NYT)* of a partition $\lambda \vdash n$ as a filling of its Young diagram with entries from an arbitrary set of integers, satisfying the same conditions as an SYT.

Row Insertion : Let $P = (P_{i,j})$ be an NYT and $k \notin P$. Then we add k to P to obtain a new NYT denoted $P \leftarrow K$ as follows:

(1) Let r be the least integer such that $P_{1,r} > k$. If no such r exists, add k to the end of row 1 and call the tableau $P \leftarrow k$
(2) If r exists, replace $k' = P_{1,r}$ by k, and call this process *bumping*. Then insert k' in row 2 like in (1)
(3) continue this way until we add a new element as the last element a (possibly empty) row to obtain the NYT $P \leftarrow k$

We use row insertion for the RSK algorithm as follows:

- Given $\pi \in S_n$, write $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ in one line notation
- We will inductively construct pairs of tableaux $(P_0, Q_0), (P_1, Q_1), \ldots, (P_n, Q_n)$ where P_i, Q_i have i cells $\forall i$, and $shape(P_i) = shape(Q_i)$
  (1) Set $(P_0, Q_0) = (\phi, \phi)$
  (2) Given (P_{i-1}, Q_{i-1}), set $P_i = P_{i-1} \leftarrow \pi_i$. Then add i to Q_{i-1} so that $shape(P_i) = shape(Q_i)$. Then $P = P_n, Q = Q_n$

Thus, P is the "insertion tableau" created by successive row insertions of entries in π, and Q is the "recording tableau" which records the changes made to the insertion tableau, in each implementation of the RSK algorithm.

Theorem 3.8. *The RSK algorithm gives a bijection between S_n and the set of pairs (P, Q) of SYT of the same shape $\lambda \vdash n$*

This can be proven be defining the inverse, i.e, $(P, Q) \mapsto \pi$. We can do this by going in the backwards direction of the RSK algorithm by an "inverse bumping" procedure and thus constructing the permutation in reverse order.

Definition 3.9. A *column-strict plane partition (CoSPP)* is a plane partition in which the non-zero entries decrease strictly along columns.

The *shape sh(P)* of a CoSPP P is the partition, the Young diagram of which it is a filling of.

3.1.2. *RSK' algorithm.* We now want to define an analog of the RSK algorithm to obtain a bijection between rectangular matrices with non-negative entries, and pairs of CoSPPs. To do this, we first define a new row insertion algorithm as follows:

Row Insertion : Given a CoSPP $P = (P_{i,j}), k \geqslant 1$, the CoSPP $P \leftarrow k$ is constructed as follows:

(1) Let r be the smallest entry such that $k > P_{1,r}$. If no such r exists, append k to the first row to obtain $P \leftarrow k$
(2) If such an r does exist, replace $k' = P_{1,r}$ by k in the previously described "bumping" manner, and row insert k' into row 2 in the same way
(3) continue this way until we add a new element as the last element a (possibly empty)row to obtain the CoSPP $P \leftarrow k$

Two line representation of a matrix : Each rectangular matrix A with non-negative integer entries has a unique 2-line representation:

$$A = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ v_1 & v_2 & \dots & v_n \end{pmatrix}$$

satisfying

a. $u_1 \geqslant u_2 \geqslant \dots \geqslant u_n$
b. if $i < j$ and $u_i < u_j$, then $v_i \geqslant v_j$
c. no. of columns in $A = \binom{i}{j} = a_{ij}$ in the two line representation of A

We use row insertion for the RSK' algorithm as follows:

- Given an $r \times s$ matrix A with non-negative integer entries, write A in two line notation
- We will inductively construct pairs of CoSPPs $(P_0, Q_0), (P_1, Q_1), \dots, (P_n, Q_n)$ where P_i, Q_i have i cells $\forall i$, and $sh(P_i) = sh(Q_i)$
  (1) Set $(P_0, Q_0) = (\phi, \phi)$
  (2) At the i^{th} stage given (P_{i-1}, Q_{i-1}), set $P = P_{i-1} \leftarrow v_i$ and add u_i to Q_{i-1} such that $sh(P_i) = sh(Q_i)$

Note that equal entries of Q are inserted from left to right.

Lemma 3.10. *The correspondence $A \xrightarrow{RSK'} (P, Q)$ is a bijection from the set of $r \times s$ matrices with non-negative integer entries to the set of pairs of CoSPPs of the same shape such that the largest part of P is at most s and the largest part of Q is at most r.*

We can construct a single plane partition from a pair of CoSPPs.

Definition 3.11. Let λ be a partition with distinct parts (strict partition). Then, the *strict shape/ shifted diagram* of λ is obtained by left-justifying the rows of its Young diagram in a staircase shape.

Let λ, μ be strict partitions with $l(\lambda) = \lambda(\mu)$, where $l(\lambda) =$ no. of non-zero parts of λ. Construct a new partition $\rho \equiv \rho(\lambda, \mu)$ by merging the strict shape of λ, and the conjugate of the strict shape of μ along the staircase shaped diagonal. Merging thus consists of deleting the $l(\lambda)$ leftmost cells in the shifted Young diagram of λ, and fitting the Young diagram of the conjugate of μ along the staircase-shaped cavity appropriately, as shown.

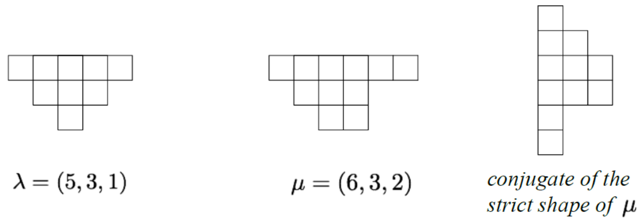


FIGURE 4.

Definition 3.12. The *rank* of a partition λ is the largest j such that $\lambda_j \geqslant j$

Lemma 3.13. *The map $(\lambda, \mu) \mapsto \rho(\lambda, \mu)$ is a bijection between pairs of strict partitions with $l(\lambda) = k$ and partitions with rank k. Then,*

$$|\rho| = |\lambda| + |\mu| - l(\lambda)$$

Proof. If $l(\lambda) = l(\mu) = k$, then merging takes place along the k leftmost cells of the shifted Young diagram of λ to form ρ. Splitting ρ along the top left diagonal of length k, results in the original two strict partitions, hence the merging process is a bijection between pairs of strict partitions of length k, and partitions with rank k. Since merging requires the deletion of $l(\lambda)$ leftmost cells and subsequent addition of $l(\mu)$ cells, we get

$$|\rho| = |\lambda| + |\mu| - l(\lambda)$$

□

Given CoSPPs P, Q of the same shape, we can thus apply ρ to each column of P and Q to form columns of a plane partition $\pi \equiv \pi(P, Q)$. Define the *conjugate* π' of π as the partition obtained by replacing the i^{th} row of π by its conjugate.

Lemma 3.14. *π is a plane partition*

For a CoSPP P, let

$$\begin{aligned} |P| &= \sum P_{i,j}, \\ \mu(P) &= \text{no. of parts of } P\text{, and} \\ max(P) &= max(P_{i,j}). \end{aligned}$$

And for a plane partition π, let

$$\begin{aligned} |\pi| &= \sum \pi_{i,j}, \\ col(\pi) &= \text{no. of columns of } \pi\text{, and} \\ row(\pi) &= \text{no. of rows of } \pi \end{aligned}$$

Note that $max(P) = max(\pi) = row(\pi')$, $max(Q) = row(\pi) = col(\pi')$ and $|P| + |Q| - \nu(P) = |\pi| = |\pi'|$.

Theorem 3.15. *Let $pp_{r,s}(n)$ denote the number of plane partitions of n with at most r rows and s columns. Then,*

$$\sum_{n \geqslant 0} pp_{r,s}(n) q^n = \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1}{1 - q^{i+j-1}}$$

Proof. Let $A_{r\times s}$ be a matrix with entries in $\mathbb{N}$. Perform the RSK' algorithm on A to obtain a pair of CoSPPs (P, Q), and further obtain the corresponding plane partition $\pi(A) = \pi(P, Q)$ by the previously described process.

In a given column in the the two-line representation of A, let i be the entry in the first row and j be the entry in the second row. We know that each such column occurs $a_{i,j}$ times, and inserts j in P and i in Q. Therefore, we have

$$\begin{aligned} |P| &= \sum_{i,j} a_{i,j} * j \\ max(P) &= max\{j | a_{i,j} \neq 0\} \end{aligned}$$

and,

$$\begin{aligned} |Q| &= \sum_{i,j} a_{i,j} * i \\ max(Q) &= max\{i | a_{i,j} \neq 0\} \end{aligned}$$

$$\implies |\pi(A)| = |P| + |Q| - \mu(P) = \sum_{i,j} a_{i,j}(i + j - 1)$$

By definition, for π a plane partition with at most r rows and s columns, we have

$$\begin{aligned} \sum_{n\geqslant 0} pp_{r,s}(n)q^n &= \sum_{\pi} q^{|\pi|} \\ &= \sum_{A_{r\times s}} q^{\sum\limits_{i,j}^{r,s} a_{i,j}(i+j-1)} \\ &= \prod_{i=1}^{r}\prod_{j=1}^{s}\left(\sum_{a_{i,j}\geqslant 0} q^{a_{i,j}(i+j-1)}\right) \\ &= \prod_{i=1}^{r}\prod_{j=1}^{s}\left(\frac{1}{1-q^{i+j-1}}\right) \end{aligned}$$

□

Let $pp_r(n)$ = number of plane partitions of n with at most r rows.

Corollary 3.16.

(1) $\sum\limits_{n\geqslant 0} pp_r(n)q^n = \prod\limits_{k\geqslant 1} \frac{1}{(1-q^k)^{min(k,r)}}$

(2) $\sum\limits_{n\geqslant 0} pp(n)q^n = \prod\limits_{k\geqslant 1} \frac{1}{(1-q^k)^k}$

The proof follows from theorem 3.15

3.2. **Descending plane partitions.**

Definition 3.17. A *descending plane partition (DPP)* is a plane partition satisfying the following:

(i) entries in the array decrease weakly across rows
(ii) on successively indenting the left edges of the array in a staircase form, entries decrease strictly along columns
(iii) the number of columns in any row is strictly less than the largest entry of that row
(iv) the number of columns is any row is greater than or equal to the largest entry of the row below it

7	7	6	6	3	1
	6	5	4	2	
		3	3		
			2		

FIGURE 5. A DPP of order 7

A DPP has order n if each entry in the array is less than or equal to n

Theorem 3.18. ***(Andrews, 1979):*** *The number of descending plane partitions of order n is equal to the number of ASMs of the same order, and is given by*

$$D_n = A_n = \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{n+i+j-1}{2i+j-1}$$

3.3. **Totally symmetric self-complementary plane partitions.** Let π be a plane partition in an $a \times b \times c$ room, i.e, π has at most a rows, b columns, and largest entry at most c. π is *symmetric* if $\pi_{i,j} = \pi_{j,i}$. The *complement* π^c of π is the set of unit cubes that fit inside the $a \times b \times c$ box, but do not belong to π. A plane partition is *self-complementary* if it is equal to its complement.

Definition 3.19. A *totally symmetric self-complementary plane partition (TSSCPP)* is a plane partition that is both symmetric and cyclically symmetric, as well as self-complementary.

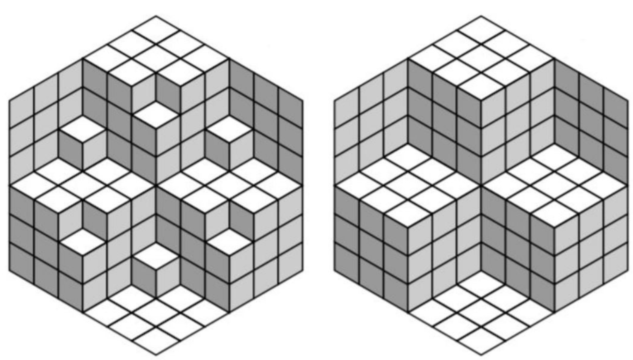


FIGURE 6. Two TSSCPPs in a $6 \times 6 \times 6$ box

Theorem 3.20. ***(Andrews, 1994):*** *The number of TSSCPPs in a $2n \times 2n \times 2n$ box equal to the number of ASMs of order n*

4. MacMahon's Box Formula

4.1. **Plane partitions as lozenge tilings.** A dimer covering, or a perfect matching of a graph G is a collection of edges that covers all the vertices exactly once, that is, each vertex is the endpoint of a unique edge.

Kasteleyn showed how to count the number of dimer coverings of an $m \times n$ square grid, and later on any planar graph. The statement is particularly simple when G is a subgraph of the honeycomb graph H of the regular tiling of the plane by hexagons, bounded by a simple polygon. Then the number of coverings Z_G is the square root of the determinant of the adjacency matrix of G.

This is relevant because another way of viewing plane partitions in an $a \times b \times c$ box, is as lozenge tilings of an $a \times b \times c \times a \times b \times c$ hexagonal region of a triangular grid. Therefore, counting the number of plane partitions boils down to counting the number of perfect matchings of the dual graph.

Theorem 4.1. ***(Kasteleyn's theorem):*** *Let G be a planar graph. Then,*

(i) there exists an orientation of G such that every face in G has an odd number of clockwise oriented edges
(ii) if $A(G)$ is the adjacency matrix for such an orientation, the number of perfect matchings in G is $\sqrt{det(A(G))}$

However, proving MacMahon's box theorem this way turns out to be difficult.

4.2. **Plane partitions as lattice paths.** A plane partition may be represented as a family of plane partitions as follows: let P be a plane partition inside an $a \times b \times c$ box, with at most a rows, at most b columns, and largest entry at most c. Each row of P is itself a partition contained within a $b \times c$ box, and is uniquely represented as an up-right path from $(b, 0)$ to $(0, c)$ as demonstrated.

Example 4.2.

$$P = \begin{matrix} 4 & 3 & 3 & 2 \\ 3 & 2 & & \\ 2 & 2 & & \\ 2 & 1 & & \end{matrix}$$

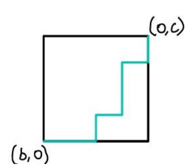


Figure 7. The partition (4,3,3,2) as a lattice path

A family of lattice paths can thus be created, with each lattice path representing each P_i:

(a) in $\mathbb{Z}^2$, the lattice path P_i starts at $(1 - i, i - 1)$ and ends at $(c + 1 - i, b + i - 1)$
(b) each entry $P_{i,j}$ indicates a vertical edge at $i - 1 + P_{i,j}$ on the x axis.

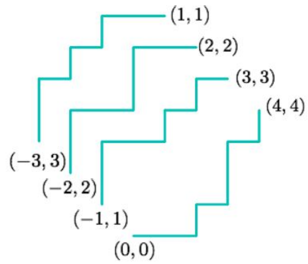


FIGURE 8. The above plane partition as family of NILPs

Notice that since $P_{i,j} \geqslant P_{i+1,j}$, we get a family of non-intersecting lattice paths (NILPs). Hence, in order to count the number of partitions in a box, we must count certain families of NILPs. The Lindström–Gessel–Viennot lemma is a way to do that.

Definition 4.3. In a permutation σ if $\sigma(i) > \sigma(j)$ for $i < j$, then $(\sigma(i), \sigma(j))$ is an *inversion pair* of σ

Definition 4.4. The *sign of a permutation* σ can be expressed as

$$sgn(\sigma) = (-1)^{inv(\sigma)}$$

where $inv(\sigma)$ is the inversion number, i.e, the number of inversion pairs in σ.

Theorem 4.5. *The inversion number $inv(\sigma)$ of a permutation $\sigma \in S_n$ has the same parity as the number of transpositions $t(\sigma)$ in the decomposition of σ, and hence*

$$sgn(\sigma) = (-1)^t$$

Proof. In the case of the identity permutation, the proof is trivial. Since every permutation can be obtained be sequentially left-multiplying the identity with transpositions, it is sufficient to show that multiplying by one transposition changes the number of inversion pairs by an odd number, hence switching the parity of $inv(\sigma)$.

Suppose we have a permutation σ. Left-multiply by $(\sigma_i\sigma_j)$ where $i < j$ to obtain $(\sigma_i\sigma_j) \cdot \sigma = \sigma'$. This clearly makes/removes the (i, j) inversion pair, changing $inv(\sigma)$ by ± 1. Next, we show that the remaining change in $inv(\sigma)$ is by an even amount.

For any $a \notin [i, j]$, it is not hard to see that the inversion status of σ_a and σ_i (or σ_j) are the same under both σ and σ'. On the other hand, if $a \in (i, j)$, then the inversion status of σ_i and σ_a flips when we go σ to σ' (and symmetrically between σ_j and σ_a). So, suppose there are N elements strictly between i and j, and K of them form inversion pairs with i (or σ_i) under σ, and hence $N - K$ did not. Under σ', we have then lost K inversion pairs, and gained $N - K$, for a net change of $N - 2K$ in $inv(\sigma)$. Symmetrically, if L elements in (i, j) formed inversion pairs with j under σ, we will have a net change of $N - 2L$ in $inv(\sigma)$.

Therefore, total change in $inv(\sigma)$ amounts to $1 + (N - 2K) + (N - 2L)$, which is an odd integer. ☐

4.3. **Lindström–Gessel–Viennot lemma.** Let $G = (V, E)$ be a finite directed acyclic graph. Let G be edge-weighted, such that the weight of a directed path $p = (v_1, \ldots, v_k)$ is the product of the weights of the contained edges:

$$w(p) = \prod_{i=1}^{k-1} w(v_i, v_{i+1})$$

Let $\mathcal{P} = (p_1, \ldots, p_n)$ be the family of paths in G with distinct starting points in $\{s_1, \ldots, s_n\}$ and distinct ending points in $\{e_1, \ldots, e_n\}$. Define the weight of the family of paths as

$$w(\mathcal{P}) = \prod_{i=1}^{n} w(p_i)$$

If the the i^{th} path starts at s_i and ends at e_{σ_i} $\forall i$ where $\sigma \in S_n$, then the sign of the family $\mathcal{P}$ is $\operatorname{sgn}(\mathcal{P}) = \operatorname{sgn}(\sigma)$. $\mathcal{P}$ is said to be *non-intersecting* if no pair of path in $\mathcal{P}$ has a vertex in common.

Theorem 4.6. ***(Karlin-McGregor, 1959; Lindström, 1973; Gessel-Viennot, 1985):*** *Let G be an edge-weighted, finite, directed, acyclic graph with two families of disjoint vertices $S = \{s_1, \ldots, s_n\}$, and $E = \{e_1, \ldots, e_n\}$. Let $p(i, j)$ be the weighted sum from s_i to e_j. Then the signed weighted enumeration of non-intersecting lattice paths (NILPs) $\mathcal{P}$ starting at $s_1, \ldots, s_n$ and ending at $e_1, \ldots, e_n$ is given by*

$$\sum_{\mathcal{P} \text{ an NILP}} sgn(\mathcal{P}) w(\mathcal{P}) = det(p(i, j))_{1 \leqslant i,j \leqslant n}$$

Proof. Using the Leibniz notation of the determinant, we get

$$det(P_{i,j}) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} p(i, \sigma_i)$$

Further expansion of the above expression results in the signed sum of weights of every possible family of paths from S to E. To isolate those terms in the summation that correspond to NILPs, we need to find a **weight-preserving, sign-reversing** involution on the set of all families, which will pairwise cancel the contribution of families with intersecting paths.

In any family $\mathcal{P} = (p_1, \ldots, p_n)$, let a be the smallest integer such that the path $p_a = (s_a = a_1, a_2, \ldots, a_n = e_{\sigma_a})$ from s_a to e_{σ_a} has an intersection. Let b be the smallest integer such that $p_b = (s_b = b_1, b_2, \ldots, b_m = e_{\sigma_b})$ and p_a intersect, and let $k = a_u = b_v$ be the last point of intersection. Interchange the vertices of p_a and p_b after u, to obtain

$$\begin{aligned} p'_a &= (s_a = a_1, a_2, \ldots, k, \ldots, b_{m-1}, b_m = e_{\sigma_b}) \\ p'_b &= (s_b = b_1, b_2, \ldots, k, \ldots, a_{n-1}, a_n = e_{\sigma_a}) \end{aligned}$$

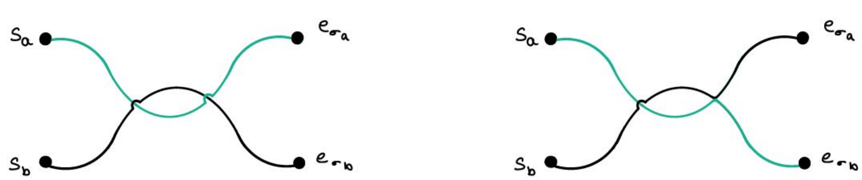


Figure 9. The LGV involution

We thus obtain another family $\mathcal{P}' = (p_1, \ldots, p_{a-1}, p'_a, p_{a+1}, \ldots, p_{b-1}, p'_b, p_{b-1}, \ldots, p_n)$, where a path starting at s_i ends at $e_{\sigma'_i}$, where $\sigma' = (\sigma_a \sigma_b) \cdot \sigma$ to account for the interchange of the endpoints of paths starting at s_a and s_b. Performing this process on $\mathcal{P}'$, we get back $\mathcal{P}$, therefore this process is an involution. Since the edges are preserved in the involution, so are the weights of the paths and thus $w(\mathcal{P}') = w(\mathcal{P})$. Also, since σ' can be obtained from σ by a single transposition, $(-1)^{t(\sigma)} = -(-1)^{t(\sigma')}$, and hence $sgn(\mathcal{P}') = -sgn(\mathcal{P})$. The involution is therefore weight-preserving and sign-reversing as desired, and contributions of families with intersecting paths cancel pairwise, proving the result. □

Corollary 4.7. ***(Simplest form of the LGV lemma)*** *Suppose G is such that the only NILPs are from s_i to $e_i \forall i$. Then, $det(P_{(i,j)})$ gives the weighted sum over all NILPs, and is manifestly positive.*

In order to apply the LGV lemma to count the families of NILPs that represent partitions in the box, we first need to find a weight-preserving sign-reversing involution on the families of NILPs. The LGV lemma cannot be used directly since as per MacMahon's formula since the weight of each lattice path is the area of the Young diagram of the partition it represents.

Definition 4.8. The *minimal path* from s_i to e_j is the path that consists of all up steps initially, followed by all right steps.

Let $ar(p(i,j))$ be the area enclosed by the path $p(i,j)$ and the minimal path from i to j.

Lemma 4.9. ***(Krattenhaler's lemma):*** *Given indeterminates $x_1, \ldots, x_n$, $a_2, \ldots, a_n$, and $b_2, \ldots, b_n$, we have*

$$det((x_i + a_n) \cdots (x_i + a_{j+1})(x_i + b_j) \cdots (x_i + b_2))_{1 \leqslant i,j \leqslant n} = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j) \prod_{2 \leqslant i \leqslant j \leqslant n} (b_i - a_j)$$

Theorem 4.10. ***(MacMahon's "Box Formula"):*** *The generating function of the number of plane partitions contained in an $a \times b \times c$ box is given by*

$$\sum_{\pi \subset a \times b \times c} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

Proof. Let $p(s_1, e_1)$ and $p(s_2, e_2)$ be two intersecting lattice paths in a family of lattice paths. Performing the LGV involution, we observe that the weights of the paths are not preserved.

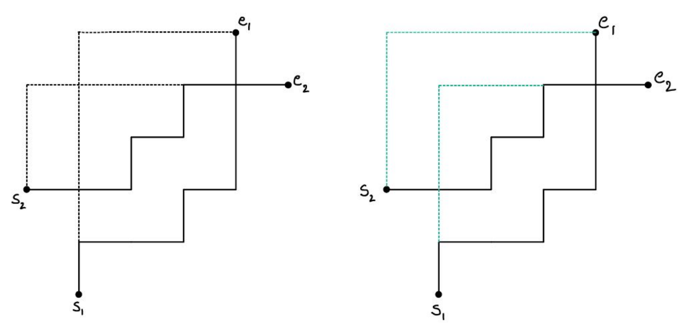


FIGURE 10. LGV involution on lattice paths

If the $x-$coordinates of s_1 and s_2 are i_1 and i_2 respectively, and the $y-$coordinates of e_1 and e_2 are j_1 and j_2 respectively, then the change in weight is

$$Ar(p(s_1, e_2)) \cdot Ar(p(s_2, e_1)) - Ar(p(s_1, e_1)) \cdot Ar(p(s_1, e_2)) \quad = \quad (i_1 - i_2)(j_1 - j_2)$$

To make the difference in weights 0, we modify the weight of each path by introducing a quantity whose contribution to the difference is $-(i_1 - i_2)(j_1 - j_2)$.

$$\begin{aligned} -(i_1 - i_2)(j_1 - j_2) &= -i_1(j_1 - j_2) + i_2(j_1 - j_2) + (i_1^2 - i_1^2) + (i_2^2 - i_2^2) \\ &= i_1(i_1 - j_1) + i_2(i_2 - j_2) - i_1(i_1 - j_2) - i_2(i_2 - j_1) \end{aligned}$$

Thus, if $w(p(i,j)) = q^{i(i-j)+Ar((p(i,j)))}$, then

$$w(p(s_1, e_1)) \cdot w(p(s_2, e_2)) = w(p(s_1, e_2)) \cdot w(p(s_2, e_1))$$

The involution from the LGV lemma is now weight-preserving and sign reversing. The quantity $i(i-j)=0$ for NILPs since $i=j$, and the path from s_i to e_i has weight q^{π_i} as wanted. Thus, by the LGV lemma,

$$\sum_{\pi \subset a\times b\times c} q^{|\pi|} = det\left(q^{i(i-j)}\begin{bmatrix}\mathbf{b+c}\\ \mathbf{b+j-i}\end{bmatrix}_{\mathbf{q}}\right)_{1\leqslant i,j\leqslant a}$$

Evaluating the determinant: Let M be the $n\times n$ matrix given by $M=\left(q^{i(i-j)}\begin{bmatrix}\mathbf{b+c}\\ \mathbf{b+j-i}\end{bmatrix}_{\mathbf{q}}\right)$.

The $(i,j)^{th}$ term of the matrix is:

$$\begin{aligned}
q^{i(i-j)}\begin{bmatrix}\mathbf{b+c}\\ \mathbf{b+j-i}\end{bmatrix}_{\mathbf{q}} &= q^{i(i-j)}\frac{[\mathbf{b+c}]_{\mathbf{q}}!}{[\mathbf{b+j-i}]_{\mathbf{q}}![\mathbf{c+i-j}]_{\mathbf{q}}!}\\
&= q^{i(i-j)}\frac{\prod_{k=1}^{b+c} 1-q^k}{\prod_{l=1}^{b+j-i} 1-q^l \prod_{m=1}^{c+i-j} 1-q^m}\\
&= q^{i(i-j)}\frac{\prod_{k=1}^{b+c} 1-q^k}{\prod_{l=1}^{b+a-i} 1-q^l \prod_{m=1}^{c+i-1} 1-q^m} \times \prod_{n=b+j-i+1}^{b+a-i} 1-q^n \prod_{p=c+i-j+1}^{c+i-1} 1-q^p
\end{aligned}$$

We factor out the $j-$independent term from each row of the determinant to get:
(4.1)

$$det(M)=\prod_{i=1}^{a}\left(\frac{\prod_{k=1}^{b+c} 1-q^k}{\prod_{l=1}^{b+a-i} 1-q^l \prod_{m=1}^{c+i-1} 1-q^m}\right)\cdot det\left(q^{i(i-j)}\prod_{k=1}^{a-j} 1-q^{b+j-i+k}\prod_{l=1}^{j-1} 1-q^{c+i-j+l}\right)$$

Consider the $(i,j)^{th}$ term of this simplified determinant. Factor out q^{-i} from each of the $(a-j)$ terms in the first product, and $-q^{j-c-l}$ from each of the $(j-1)$ terms in the second product, to get:

$$\begin{aligned}
(4.2)\quad det&\left((-1)^{(j-1)}q^{i(i-j)-i(a-j)+c(j-1)-\binom{j}{2}}\prod_{k=1}^{a-j} q^i - q^{b+j-i+k}\prod_{l=1}^{j-1} q^i - q^{j-c-l}\right)\\
&= \prod_{i=1}^{i=a}(-1)^{(i-1)}q^{i^2-ia+c(i-1)-\binom{i}{2}}\cdot det\left(\prod_{k=1}^{a-j} q^i - q^{b+j-i+k}\prod_{l=1}^{j-1} q^i - q^{j-c-l}\right)
\end{aligned}$$

We can now use Krattenhaler's lemma (lemma 4.9) to further simplify this determinant to get:

$$\begin{aligned}
(4.3)\quad det&\left(\prod_{k=1}^{a-j} q^i - q^{b+j-i+k}\prod_{l=1}^{j-1} q^i - q^{j-c-l}\right) = \prod_{1\leqslant i<j\leqslant a}(q^i-q^j)\prod_{2\leqslant i\leqslant j\leqslant a}(-q^{-c+i-1}+q^{b+j})\\
&= \prod_{i=1}^{a} q^{i(a-i)}\prod_{1\leqslant i<j\leqslant a}(1-q^{j-i})\prod_{i=2}^{a} q^{(-c+i-1)(a+1-i)}\prod_{2\leqslant i\leqslant j\leqslant a}(1-q^{b+c+j-i})
\end{aligned}$$

Combining 4.1, 4.2 and 4.3, we get

$$(4.4)\quad det(M) = \prod_{i=2}^{a} q^{i(c-a)-c+\binom{i+1}{2}+i(a-i)+(-c+i-1)(a+1-i)}$$
$$\times \prod_{i=1}^{a}\left(\frac{\prod\limits_{k=1}^{b+c} 1-q^k}{\prod\limits_{l=1}^{b+a-i} 1-q^l \prod\limits_{m=1}^{c+i-1} 1-q^m}\prod_{i=1}^{a} q^{i(a-i)} \prod_{1\leqslant i<j\leqslant a}(1-q^{j-i}) \prod_{2\leqslant i\leqslant j\leqslant a}(1-q^{b+c+j-i})\right)$$

The pre-factor of q is:
$\sum\limits_{i=2}^{a}[i(c-a)-c+\binom{i+1}{2}+i(a-i)+(-c+i-1)(a+1-i)]$

$$\begin{aligned}
&= \sum_{i=2}^{a}[2ic-2c-\frac{3i^2}{2}+\frac{5i}{2}-ac+ai-a-1] \\
&= -(ac+2c+a+1)(a-1)+(2c+a+\frac{5}{2})(\frac{a(a+1)}{2}-1)-\frac{3}{2}[\frac{a(a+1)(2a+1)}{6}-1] \\
&= 0
\end{aligned}$$

The remaining factors can be written as:

$$\begin{aligned}
&= \prod_{i=1}^{a}\left(\frac{\prod\limits_{k=1}^{b+c}(1-q^k)\prod\limits_{i=1}^{a} q^{i(a-i)} \prod\limits_{j=i+1}^{a}(1-q^{j-i})\prod\limits_{j=i}^{a}(1-q^{b+c+j-i+1})}{\prod\limits_{l=1}^{b+a-i}(1-q^l)\prod\limits_{m=1}^{c+i-1}(1-q^m)}\right)\times\prod_{j=1}^{a}\frac{1}{1-q^{b+c+j}} \\
&= \prod_{i=1}^{a}\left(\frac{\prod\limits_{k=1}^{b+c}(1-q^k)\prod\limits_{i=1}^{a} q^{i(a-i)} \prod\limits_{j=1}^{a-i}(1-q^{j})\prod\limits_{j=1}^{a-i+1}(1-q^{b+c+j})}{\prod\limits_{l=1}^{b+a-i}(1-q^l)\prod\limits_{m=1}^{c+i-1}(1-q^m)}\right)\times\prod_{j=1}^{a}\frac{1}{1-q^{b+c+j}} \\
&= \prod_{i=1}^{a}\left(\frac{\prod\limits_{k=1}^{a+b+c-i+1}(1-q^k)}{\prod\limits_{l=a-i+1}^{a-l+b}(1-q^l)\prod\limits_{m=1}^{c+i-1}(1-q^m)}\right)\times\prod_{j=1}^{a}\frac{1}{1-q^{b+c+j}}
\end{aligned}$$

Replacing i by $a-i+1$ in the k and l products,

$$= \prod_{i=1}^{a}\left(\frac{\prod_{k=1}^{b+c+i}(1-q^k)}{\prod_{l=i}^{i+b-1}(1-q^l)\prod_{m=1}^{c+i-1}(1-q^m)}\right)\times\prod_{j=1}^{a}\frac{1}{1-q^{b+c+j}}$$

$$= \prod_{i=1}^{a}\left(\frac{\prod_{k=c+i}^{b+c+i-1}(1-q^k)}{\prod_{l=i}^{i+b-1}(1-q^l)}\right)$$

$$= \prod_{i=1}^{a}\prod_{j=1}^{b}\frac{1-q^{i+j+c-1}}{1-q^{i+j-1}}$$

$$= \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

Hence proven □

5. Symmetry Classes of Plane Partitions

A plane partition π is a *symmetric plane partition (SPP)* if $\pi_{i,j} = \pi_{j,i}\ \forall i,j \in \mathbb{N}$.

For convenience, define the set

$$\mathcal{B}(a,b,c) = \{(i,j,k) | 1 \leqslant i \leqslant a,\ 1 \leqslant j \leqslant b,\ 1 \leqslant k \leqslant c\}$$

so that a plane partition π is a subset of $\mathcal{B}$ such that if $(i,j,k) \in \pi$, then $(i-1,j,k), (i,j-1,k), (i,j,k-1) \in \pi$. A symmetric plane partition in $\mathcal{B}(a,a,c)$ is a partition π such that if $(i,j,k) \in \pi$ then $(j,i,k) \in \pi$.

Conjecture 5.1. ***(MacMahon's conjecture):*** *The generating function for symmetric plane partitions that are subsets of $\mathcal{B}(a,a,c)$ is given by*

$$\sum_{\substack{\pi\subseteq\mathcal{B}(a,a,c)\\ S_2(\pi)=\pi}} q^{|\pi|} = \left(\prod_{i=1}^{a}\prod_{k=1}^{c}\frac{1-q^{1+2i+k-2}}{1-q^{2i+k-2}}\right)\left(\prod_{1\leqslant i<j\leqslant a}\prod_{k=1}^{c}\frac{1-q^{2+2(i+j+k-2)}}{1-q^{i+j+k-2}}\right)$$

Ian Macdonald realised that this conjecture could be expressed differently.

Let S_2 be the group of permutations (of order 2) acting on the first two coordinates of an SPP π. Let $\mathcal{B}/S_2$ be the set of orbits of elements of $\mathcal{B}$ under the action of S_2. There are two types of orbits:

(i) singletons of the form $\{(i,i,k)\}$
(ii) doubletons of the form $\{(i,j,k),(j,i,k)\}$, where $i \neq j$

For an orbit η, let $|\eta|$ denote the size of the orbit. Define the *height* of an element (i,j,k) as

$$ht(i,j,k) = i+j+k-2$$

and let the height of an orbit, $ht(\eta)$, be the height of any of its elements. Then, MacMahon's conjecture for SPPs can be rewritten as:

$$\sum_{\substack{\pi\subseteq\mathcal{B}(a,a,c)\\ S_2(\pi)=\pi}} q^{|\pi|} = \prod_{\eta\in\mathcal{B}(a,a,c)/S_2} \frac{1-q^{|\eta|(1+ht(\eta))}}{1-q^{|\eta|ht(\eta)}}$$

This way of rewriting the conjecture further enabled Macdonald to conjecture formulae for cyclically symmetric plane partitions (CSPPs) and totally symmetric plane partitions (TSPPs).

Conjecture 5.2. ***(Macdonald's conjecture):*** *The generating function for cyclically symmetric plane partitions that are subsets of $\mathcal{B}(a,a,a)$ is given by*

$$\sum_{\substack{\pi\subseteq\mathcal{B}(a,a,a)\\ \pi \text{ a CSPP}}} q^{|\pi|} = \prod_{\eta\in\mathcal{B}(a,a,a)/C_3} \frac{1-q^{|\eta|(1+ht(\eta))}}{1-q^{|\eta|ht(\eta)}}$$

where C_3 is the cyclic group of order 3, acting on (i,j,k).

Conjecture 5.3. ***(Macdonald's TSPP conjecture):*** *The generating function for totally symmetric plane partitions that are subsets of $\mathcal{B}(a,a,a)$ is given by*

$$\sum_{\substack{\pi\subseteq\mathcal{B}(a,a,a)\\ \pi \text{ a TSPP}}} q^{|\pi|} = \prod_{\eta\in\mathcal{B}(a,a,a)/S_3} \frac{1-q^{|\eta|(1+ht(\eta))}}{1-q^{|\eta|ht(\eta)}}$$

where S_3 is the group of all permutations acting on (i,j,k).

Similarly, replacing S_2 with the trivial group, S_1, yields the number of plane partitions contained in $\mathcal{B}(a,b,c)$.

5.1. **Symmetric functions.** Let R be a commutative ring with identity and let x be an indeterminate.

Definition 5.4. The *ring of formal power series* in x denoted $R[[x]]$ consists of formal sums of the form

$$f(x) = \sum_{n\geqslant 0} a_n x^n \text{ ,where } a_n \in \mathbb{R}\ \forall\ n$$

Addition and multiplication on $R[[x]]$ is defined as on the ring of polynomials $R[x]$, with identity elements being 0 and 1 respectively. We can extend this definition to the ring of formal power series in countably infinite indeterminates, denoted $R[[x_1, x_2, x_3, ...]]$, where each summand is a monomial of finite degree.

Definition 5.5. Let $n \in \mathbb{N}$. A *weak composition* of n is an infinite sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3 \ldots)$ such that

$$\sum_{i=1}^{\infty} \alpha_i = n$$

Definition 5.6. A *homogeneous symmetric function of degree* n over R is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha \mathbf{x}^\alpha, \quad \text{where}$$

(i) α ranges over all weak compositions of n
(ii) $c_\alpha \in R$
(iii) $\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \ldots$

(iv) $f(x_1, x_2, \ldots) = f(x_{\sigma_1}, x_{\sigma_2}, \ldots)$ for every permutation σ of $\mathbb{N}$.

Let $\Lambda_R^n \equiv \Lambda^n$ be the set of homogeneous symmetric functions of degree n.

- If $f, g \in \Lambda^n$ and $a, b \in R$, then $af + bg \in \Lambda^n$. If $R = \mathbb{Q}$, Λ_R^n is a vector space.
- If $f \in \Lambda^m, g \in \Lambda^n$, then Λ^{m+n}
- If we write $\Lambda_R = \Lambda_R^0 \oplus \Lambda_R^1 \oplus \Lambda_R^2 \ldots$, then every symmetric function f can be written as $f = f^0 + f^1 + f^2 + \ldots$ where $f^i \in \Lambda_R^i$. Then, Λ_R becomes an R-algebra.

5.1.1. *Schur functions.* Let $\lambda \vdash n$. A *Semi-standard Young Tableau (SSYT)* of shape λ is a filling of the Young diagram of λ by positive integers that increase weakly along rows and increase strictly along columns. If T is an SSYT of shape λ, we write $sh(T) = \lambda$. The *type* of T denoted by $\alpha = (\alpha_1, \alpha_2, \ldots) = type(T)$ is a weak composition where $\alpha_i \equiv \alpha_i(t)$ is the number of parts of T equal to i. For an SSYT of type α we write

$$\mathbf{x}^{\mathbf{T}} = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \ldots$$

Definition 5.7. Given a partition $\lambda \vdash n$, define the *Schur function* s_λ indexed by λ, in the variables $x_1, x_2, \ldots$ as

$$s_\lambda = \sum_{\substack{T \\ sh(T)=\lambda}} \mathbf{x}^{\mathbf{T}}$$

Theorem 5.8. *Schur functions are symmetric, i.e $s_\lambda \in \Lambda$*

Proof. Any transposition $(ij), i < j$ can be written as a product of adjacent transpositions in the form $(i\ i+1)(i+1\ i+2)\ldots(j-1\ j)\ldots(i+1\ i+2)(i\ i+1)$. Since $s_\lambda(x)$ is invariant under any permutation $\mathbb{N}$ interchanging finitely many positions, it suffices to prove that s_λ is invariant under the interchange of x_i and x_{i+1}.

Let $\lambda \vdash n$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak compositions of n. Interchange α_i, α_{i+1} to get

$$\alpha' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots)$$

and define

$$\mathcal{T}_{\lambda,\alpha} = \{\text{SSYT } T | sh(T) = \lambda, type(T) = \alpha\}$$

$|\mathcal{T}_{\lambda,\alpha}|$ gives the coefficient of $\mathbf{x}^{\mathbf{T}}$, and we need to show that $|\mathcal{T}_{\lambda,\alpha}| = |\mathcal{T}_{\lambda,\alpha'}|$.

Let T $\in \mathcal{T}_{\lambda,\alpha}$. Consider those parts of T that are equal to either i or $i+1$. We find that in any column, there are three possibilities:

(i) The column contains neither i nor $i+1$
(ii) It contains both i and $i+1$
(iii) It contains only one of i and $i+1$

Since (i) is inconsequential and cardinality is preserved in (ii), we focus our our attention on (iii). The most general configuration of (iii) is :

$$\underbrace{i \ldots i}_{r_0}\ \underbrace{i \ldots i}_{r}\ \underbrace{i+1 \ldots i+1}_{s}\ \underbrace{i+1 \ldots i+1}_{s_0}$$

Here, there are r_0 $(i+1)$s under the first r_0 is in the following row, and s_0 is under the last s_0 $(i+1)$s in the previous row. All parts below the next r is will therefore be strictly greater than $i+1$, and all parts directly above the s $(i+1)$s will be strictly less than i. There thus exists a tableau $\phi(T)$, with s is and r $(i+1)$s in each such block. $\phi : \mathcal{T}_{\lambda,\alpha} \mapsto \mathcal{T}\lambda, \alpha'$ is clearly an involution, and hence proven that $|\mathcal{T}_{\lambda,\alpha}| = |\mathcal{T}_{\lambda,\alpha'}|$. $\square$

Definition 5.9. Let $\lambda \vdash n$, and α be a weak composition of n. Then, the *Kostka number* $K_{\lambda,\alpha}$ is the number of SSYTs of shape λ and weight (or type) α:

$$K_{\lambda,\alpha} = |\mathcal{T}_{\lambda,\alpha}|$$

Then, by theorem 5.8

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu$$

where μ is the partition made by rearranging α. For $\mu = \langle 1^n \rangle$, $K_{\lambda,\langle 1^n \rangle}$ is given by the hook-length formula (theorem 3.6).

Proposition 5.10. *$\{s_\lambda | \lambda \vdash n\}$ is a basis for Λ^n*

Proposition 5.11. *$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$, i.e, $\{s_\lambda\}$ forms an orthonormal basis for Λ*

Alternate definition of Schur functions (Jacobi's bialternant formula):
Fix n variables $x_1, \ldots, x_n$. Let $\lambda \vdash n$ be a partition with $\lambda_1 > \lambda_2 > \ldots \lambda_n \geqslant 0$. Define $\delta = (n-1, n-2, \ldots, 1, 0)$. Define the skew-symmetric function $a_{\lambda+\delta}(x_1, \ldots, x_n)$ as

$$a_{\lambda+\delta} = det(x_i^{\lambda_j+n-j})_{1 \leqslant i,j \leqslant n} = \sum_{\pi \in \S_n} sgn(\pi) \prod_{k=1}^{n} x_k^{\lambda_{\pi_k}+n-\pi_k}$$

Since $a_{\lambda+\delta}$ is alternating, it is divisible by the Vandermonde determinant $= det(x_i^{n-j}) = a_\delta$, and their ratio is a symmetric polynomial in $x_1, \ldots, x_n$.

Definition 5.12. Schur polynomials in n variables $x_1, \ldots, x_n$ indexed by a partition $\lambda \vdash n$, are defined as follows:

$$s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta}$$

Theorem 5.13. ***(Cauchy Identity):*** *Let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ be two families of variables. Then for all partitions λ,*

$$\sum_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) = \prod_{i,j \geqslant 1} \frac{1}{1 - x_i y_j}$$

Theorem 5.14. ***(Jacobi-Trudi Identity):*** *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n, where $\lambda_1 \geqslant \ldots \geqslant \lambda_k \geqslant 0$. Schur polynomials can be expressed in terms of complete homogeneous polynomials as follows:*

$$s_\lambda(x_1, \ldots, x_n) = det(h_{\lambda_j - j + i}(x_1, \ldots, x_n))_{1 \leqslant i,j \leqslant k}$$

Proof. Let $T_{n,\lambda}$ be an SSYT of shape λ with entries in $[n]$. Each $T \in T_{n,\lambda}$ can be expressed as a family of non-intersecting lattice paths as follows: the i^{th} lattice path goes from $s_i = (1, k-i)$ to $e_i = (n, k_{\lambda_j} - j)$, taking up and right steps. Assign each horizontal edge the weight 1, and the vertical edge in column j, the weight x^j. For example, the SSYT

$$\begin{array}{ccccc} 1 & 1 & 1 & 2 & 4 \\ 2 & 3 & 4 & & \\ 3 & 4 & 5 & & \\ 4 & 5 & & & \end{array}$$

has weight $x_1^3 x_2^2 x_3^2 x_4^4 x_5^2$. Since each SSYT has strictly increasing entries along the columns, the corresponding family of lattice paths is non-intersecting. There is thus a bijection between $T_{n,\lambda}$ and families of NILPs with specific starting and ending points.

Now, since a path from s_i to e_j has $\lambda_j - j + i$ up steps, the degree of its weight is $\lambda_j - j + i$. Moreover, the sum of the weights of all paths from $s_i \to e_j$ is a symmetric function of $x_1, \ldots, x_n$. Since every monomial has coefficient 1 and degree at most n,

$$wt(s_i \to e_j) = h_{\lambda_j - j + i}(x_1, \ldots, x_n)$$

Using the LGV lemma, we now have

$$s_\lambda(x_1, \ldots, x_n) = det(wt(s_i \to e_j))_{1 \leqslant i,j \leqslant k} = det(h_{\lambda_j - j + i}(x_1, \ldots, x_n))_{1 \leqslant i,j \leqslant k}$$

□

We can also use Schur functions to prove Macmahon's Box formula.

5.2. **Alternate proof of the Box formula.** We can write any semi-standard Young tableau in terms of variables by replacing the entry with value k by x_k. For example, the previously defined SSYT may be written as

$$\begin{array}{lllll} x_1 & x_1 & x_1 & x_2 & x_4 \\ x_2 & x_3 & x_4 & & \\ x_3 & x_4 & x_5 & & \\ x_4 & x_5 & & & \end{array}$$

We can replace the x_is with powers of q to represent plane partitions in a box. For example, setting $x_1 = q^6$ would mean that each x_1 represents a stack of 6 cubes. However, since SSYTs are strictly decreasing along columns, we can only obtain CoSPPs in this manner, with weight equal to the sum of the entries $= \mathbf{x}^{\mathbf{T}} =$ weight of the SSYT.

Suppose $\lambda = (b^a)$, and T is an SSYT of shape λ. To get a plane partition, subtract $a + 1 - i$ from each entry in the i^{th} row. Setting $x_1 = q^{a+c}, \ldots, x_{a+c-1} = q^2, x_{a+c} = q$, and subtracting $a + 1 - i$ from each entry means multiplying the weight of the CoSPP by $q^{-b-2b-\ldots-ab} = q^{-ba(a+1)/2}$ to the weight of the corresponding plane partition in an $a \times b \times c$ box. Since any plane partition inside $\mathcal{B}(a, b, c)$ can be obtained in this manner, we get

$$\sum_{\pi \subseteq a \times b \times c} q^{|\pi|} = q^{-ba(a+1)/2} s_\lambda(q^{a+c}, q^{a+c-1}, \ldots, q)$$

which can be evaluated using Jacobi's bialternant formula (5.12).

5.3. **Proof of MacMahon's SPP Conjecture.** MacMahon conjectured that the generating function for symmetric plane partitions in an $a \times a \times c$ box, i.e, SPPs that are subsets of $\mathcal{B}(a, a, c)$, is as follows:

$$\prod_{\substack{\pi \subseteq \mathcal{B}(a,a,c) \\ S_2(\pi) = \pi}} q^{|\pi|} = \left(\prod_{i=1}^{a} \prod_{k=1}^{c} \frac{1 - q^{1+2i+k-2}}{1 - q^{2i+k-2}} \right) \left(\prod_{1 \leqslant i < j \leqslant a} \prod_{k=1}^{c} \frac{1 - q^{2+2(i+j+k-2)}}{1 - q^{i+j+k-2}} \right)$$

which can be written as

$$\prod_{\eta \in \mathcal{B}(a,a,c)/S_2} \frac{1 - q^{|\eta|(1+ht(\eta))}}{1 - q^{|\eta| ht(\eta)}}$$

The proof of MacMahon's conjecture can be divided into three steps: the first step is to express the generating function for SPPs as Schur functions. Then, we use Macdonald's formula to sum all Schur functions whose partitions fir inside an $a \times c$ rectangle. Lastly, we use the Weyl denominator formula to simplify the resulting determinants as products.

Lemma 5.15.

$$\sum_{\substack{\pi \, an \; SPP \\ \pi \subseteq a \times a \times c}} = \sum_{\lambda \subseteq (c^a)} s_\lambda(q^{2a-1}, q^{2a-3}, \ldots, q^3, q)$$

Proof. There is a bijection between SPPs of n in an $a \times a \times c$ box and CoSPPs of n with odd heights in an $a \times c \times (2a-1)$ box. This can be shown by first slicing an SPP parallel to the xy plane into "levels", i.e subsets of cubes of the same heights. Since each level is symmetric about the $x = y$ plane, we can further decompose them into the hooks of the cubes lying in the $x = y$ plane. Because of the symmetry in each level, hook lengths are odd and strictly decreasing within any particular level. We can reassemble the hooks into columns such that the i^{th} stack in the j^{th} column is equal to the i^{th} hook at the j^{th} level, and $i \leqslant a$, $j \leqslant c$ and each hook contains at most $2a-1$ cubes. We end up with a CoSPP in an $a \times c \times (2a-1)$ box, where each non-zero entry is odd. This process is reversible and weight-preserving, i.e, the number of cubes is preserved.

This set of CoSPPs is in bijection with semi-standard Young tableaux inside (c^a). Adding 1 to each entry in a CoSPP and dividing by 2 results in an an SSYT inside an $a \times c \times a$ box. Then, by the same kind of bijection described in the alternate proof of the Box Theorem, the generating function of CoSPPs is

$$\sum_{\lambda \subseteq (c^a)} s_\lambda(q^{2a-1}, \ldots, q^3, q)$$

□

Recall that we had defined $s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta}$. This is a special case of the *Weyl character formula*. This formula for Schur functions is the character of the group $GL_n(\mathbb{C})$ for the representation indexed by λ, and thus Schur functions are also called GL_n characters.

Similarly, there are characters formulae for other classical matrix groups.

Definition 5.16. The *orthogonal group* is

$$O(n) = \{A \in M_n | AA^* = I\}$$

where $*$ is used to represent the summation conjugate.

Definition 5.17. The *odd orthogonal characters* denoted SO^{odd}_λ, indexed by either a partition of a half-partition (i.e a partition with entries in $\mathbb{N} + 1/2$) $\lambda = (\lambda_1, \ldots, \lambda_n)$ is given by

$$SO^{odd}_(x_1, \ldots, x_n) = \frac{det\left(x_i^{\lambda_j+n-j+1/2} - x_i^{-\lambda_j-n+j-1/2}\right)_{1 \leqslant i,j \leqslant n}}{det\left(x_i^{n-j+1/2} - x_i^{-n+j-1/2}\right)_{1 \leqslant i,j \leqslant n}}$$

Theorem 5.18. ***(Weyl denominator identity):*** *the identity states that*

$$det\left(x_i^{n-j+1/2} - x_i^{-n+j-1/2}\right)_{1 \leqslant i,j \leqslant n} = \prod_{i=1}^{n} (x_i^{1/2} - \bar{x}_i^{1/2}) \prod_{1 \leqslant i,j \leqslant n} \frac{(x_i - x_j)(x_i x_j - 1)}{x_i x_j}$$

where $\bar{x} = \frac{1}{x}$

Remark 5.19. As in the GL_n case, this is symmetric in $x_i \leftrightarrow x_j$, and in addition, there is the $x_i \leftrightarrow \bar{x}_i$, $\forall i$. Unlike the GL_n case, this is a Laurent polynomial.

Lemma 5.20.

$$\sum_{\lambda \subseteq (c^a)} s_\lambda(x_1, \ldots, x_a) = SO^{odd}_{((\frac{c}{2})^a)}(x_1, \ldots, x_a) \cdot (x_1, \ldots, x_a)^{c/2}$$

By lemma 5.15 and lemma 5.20, we have

$$\sum_{\substack{\pi \text{ an } SPP \\ \pi \in a\times a\times c}} q^{|\pi|} = SO^{odd}_{(\frac{c}{2})^a}(x_1,\dots,x_a) * (x_1x_2\dots x_a)^{c/2}$$

and we need to evaluate the character for $x_i = q^{2a-2i+1}$.
Denominator: The denominator of the character is

$$\begin{aligned} &det\left(q^{2a-2i+3/2+a-j} - \bar{q}^{2a-2i+3/2+a-j}\right) \\ &\qquad = \prod_{i=1}^{a}\left(q^{(2a+1-2i)/2} - q^{-(2a+1-2i)/2}\right)\prod_{1\leqslant i<j\leqslant a}\frac{\left(q^{2a+1-i}-q^{2a+1-j}\right)\left(q^{4a+2-i-j}-1\right)}{q^{4a+2-i-j}} \end{aligned}$$

Replace i and $a+1-i$, and j and $a+1-j$ to get

$$\begin{aligned} &= \prod_{i=1}^{a}\left(q^{(i-1/2)} - q^{-(i-1/2)}\right)\prod_{1\leqslant j<i\leqslant a}\frac{\left(q^{2i-1}-q^{2j-1}\right)\left(q^{2i+2j-2}-1\right)}{q^{2i+2j-1}} \\ &= \frac{1}{q^{1/2+3/2+\ldots+(2a-1)/2}}\prod_{i=1}^{a}\left(q^{2i-1}-1\right)\frac{\prod_{1\leqslant j<i\leqslant a}\left(q^{2i-1}-q^{2j-1}\right)\left(q^{2i+2j-2}-1\right)}{q^{2(2+\ldots+a)+2(4+\ldots+(a+1))+\ldots+2(2a-2)}} \\ &= \frac{1}{q^{a^2/2+a^2(a-1)}}\prod_{i=1}^{a}\left(q^{2i-1}-1\right)\prod_{1\leqslant j<i\leqslant a}\left(q^{2i-1}-q^{2j-1}\right)\left(q^{2i+2j-2}-1\right) \end{aligned}$$

Numerator: The numerator is of the form

$$\text{Numerator} = det\left(x_i^{c/2+a-j-1/2} - \bar{x}_i^{-c/2+a-j+1/2}\right)$$

where $x_i = q^{2a-2i+1}$.

Expand the determinant using Leibniz expansion. Since there are two terms in each entry of the matrix, there are 2^a terms corresponding to each permutation. We can further simplify the expansion by taking a subset T of $[a]$ and choosing values of j from T for the second term as follows:

$$\begin{aligned} &= \sum_{\sigma\in S_a}\sum_{T\subseteq[a]}(-1)^{sgn(\sigma)+|T|}\prod_{j\notin T}x_{\sigma_j}^{c/2+a-j+1/2}\prod_{j\in T}x_{\sigma_j}^{-c/2-a+j-1/2} \\ &= \sum_{\sigma\in S_a}\sum_{T\subseteq[a]}(-1)^{sgn(\sigma)+|T|}\prod_{j=1}^{a}x_{\sigma_j}^{\epsilon_j(c/2+a-j+1/2)} \end{aligned}$$

where

$$\epsilon_j = \begin{cases} -1 & j\in T \\ +1 & j\notin T \end{cases}$$

Interchanging σ and σ^{-1}, and ϵ_j and $\epsilon_{\sigma^{-1}(j)}$,

$$= \sum_{\sigma\in S_a}\sum_{T\subseteq[a]}(-1)^{sgn(\sigma)+|T|}\prod_{j=1}^{a}x_j^{\epsilon_j(c/2+a-\sigma_j+1/2)}$$

Substitute x_j with $q^{2a+1-2j}$:

$$= \sum_{\sigma\in S_a}\sum_{T\subseteq[a]}(-1)^{sgn(\sigma)+|T|}\prod_{j=1}^{a} q^{\epsilon_j(c/2+a-\sigma_j+1/2)(2a+1-2j)}$$

$$= \sum_{\sigma\in S_a}\sum_{T\subseteq[a]}(-1)^{sgn(\sigma)+|T|}\prod_{j=1}^{a}(q^{c+2a-2\sigma_j+1/2})^{a+1/2-j}$$

$$= det\left((q^{c+2a-i+1})^{a+1/2-j}-(q^{-c+2a-i+1})^{a+1/2-j}\right)$$

Using the Weyl denominator identity 5.18:

$$=\prod_{i=1}^{a}\left(q^{c/2+a-i+1/2}-q^{-c/2+a-i+1/2}\right)\prod_{1\leqslant i<j\leqslant a}\frac{\left(q^{c+2a-2i+1}-q^{c+2a-2j+1}\right)q^{2c+4a-2i-2j+2}-1}{q^{2c+4a-2i-2j+2}}$$

Replace i with $a+1-i$, and j with $a+1-j$

$$= \prod_{i=1}^{a}\left(q^{c/2+i-1/2}-q^{-c/2+i-1/2}\right)\prod_{1\leqslant j<i\leqslant a}\frac{\left(q^{2i-1}-q^{2j-1}\right)q^{2c+2i+2j-2}-1}{q^{c+4a-2i-2j+2}}$$

$$= \frac{\prod_{i=1}^{a}\left(q^{c+2i-1}-1\right)}{q^{a(c-1)/2+a(a+1)/2}} * \frac{\prod_{i\leqslant j<i\leqslant a}\left(q^{2i-1}-q^{2j-1}\right)\left(q^{2c+2i+2j-2}-1\right)}{q^{c\binom{a}{2}+a^2(a-1)/2}}$$

Combining everything, we get

$$= \prod_{i=1}^{a}\frac{\left(q^{c+2i-1}-1\right)}{q^{2i-1}-1}\prod_{1\leqslant j<i\leqslant a}\frac{\left(q^{2c+2i+2j-2}-1\right)}{\left(q^{2i+2j-2}-1\right)} * q^{a^2c/2} * \frac{q^{a^2/2}\cdot q^{a^2(a-1)}}{q^{ac/2+a^2/2}\cdot q^{c\binom{a}{2}+a^2(a-1)}}$$

$$= \prod_{i=1}^{a}\frac{\left(q^{c+2i-1}-1\right)}{q^{2i-1}-1}\prod_{1\leqslant j<i\leqslant a}\frac{\left(q^{2c+2i+2j-2}-1\right)}{\left(q^{2i+2j-2}-1\right)}$$

$$= \left(\prod_{i=1}^{a}\prod_{k=1}^{c}\frac{1-q^{1+2i+k-2}}{1-q^{2i-+k-2}}\right)\left(\prod_{1\leqslant i<j\leqslant a}\prod_{k+1}^{c}\frac{1-q^{2+2(i+j+k-2)}}{1-q^{2(i+j+k-2)}}\right)$$

Which is Macmahon's formula.

6. Aztec Diamonds

For an ASM A of order n, let $\mu(A)$ be the number of (-1) entries in A. Then, a 2-enumeration of ASMs is $\sum 2^{\mu(A)} = 2^{n(n+1)/2}$. In 1992, Elkies, Kuperberg, Larsen and Propp introduced a new class of objects called Aztec diamonds, and showed that an Aztec diamond of order n has $2^{n(n+1)/2}$ domino tilings. This result can be proven in two ways: by establishing a correspondence between a domino tiling of an Aztec diamond and a compatible pair of ASMs, and by weighted enumeration of monotone triangles to count the domino tilings of an Aztec diamond.

Definition 6.1. An *Aztec diamond of order n*, A_n is the union of lattice squares $[a,a+1]\times[b,b+1]$ $(a,b\in\mathbb{Z})$ lying inside the region $\{(x,y)\in\mathbb{R}^2 : |x|+|y|\leqslant n+1\}$

A tiling of a region R is a set of non-overlapping tiles whose union is R, where a tile may be any closed connected region in $\mathbb{R}^2$. A domino tiling of A_n is equivalent to a perfect matching of its dual graph. For any tiling T of A_n, and for an integer $k\leqslant n$, the line $y=k$ divides A_n into two regions with an even number of lattice squares. Then, an even number of vertical dominoes must cross $y = k$ in T, and hence the number of vertical dominoes in T must be even. Let $v(T) = \frac{1}{2} \times$ (number of vertical dominoes in T).

Let an "elementary move" be the action of rotating the 2×2 block formed by two adjacent horizontal (/vertical) dominoes in a region by 90°. Then, define the rank $r(T)$ of T, to be the minimum number of elementary moves needed to obtain T from the all horizontal tiling, assuming this is possible.

Let

$$AD(n; x, q) = \sum_T x^{v(T)} q^{r(T)}$$

where T is a domino tiling of A_n. Then,

Theorem 6.2. ***Elkies, Kuperberg, Larsen, Propp****:*

$$AD(n : x, q) = \prod_{k=0}^{n-1} (1 + xq^{2k+1})^{n-k}$$

Corollary 6.3.

(a) set $x = 1$, then

$$AD(n; q) = \prod_{k=0}^{n-1} (1 + q^{2k+1})^{n-k}$$

(b) set $q = 1$, then

$$AD(n; x) = \prod_{k=0}^{n-1} (1 + x)^{n-k} = (1 + x)^{\frac{n(n+1)}{2}}$$

(c) set $x = q = 1$, then

$$AD(n) = 2^{\frac{n(n+1)}{2}}$$

Notice that for an ASM A of order n, $\sum_A 2^{\mu(A)} = 2^{\frac{n(n+1)}{2}}$

6.1. **Height Functions.** In order to determine the rank $r(T)$ of a tiling T, we assign unique integers to each corner of each lattice square in A_n. Write G as the graph of A_n with

$$\begin{aligned} V &= \{(a, b) \in \mathbb{Z}^2 : |a| + |b| \leqslant n + 1\} \\ E &= \{((u_1, u_2), (v_1, v_2)) \in V^2 : u_1 = v_1 \pm 1 \text{ or } u_2 = v_2 \pm 1\} \end{aligned}$$

Colour the lattice squares in G alternately black and white, such that each square shares its sides with oppositely coloured squares, and the line $x + y = n + 1$ passes through only white squares. Then, orient the edges of G such that arrows circulate clockwise around black squares, and anticlockwise around white squares. Define $v = (a, b)$ to be a *boundary vertex* if $|a| + |b| = n$ or $n + 1$, and let the *boundary cycle* be the closed cycle containing all boundary vertices including $(0, \pm(n + 1))$ and $(\pm(n + 1), 0)$. A vertex $v = (a, b)$ is *even* if it is the top left corner of a white square, i.e, $a + b + n + 1$ is an even integer.

Note that there are four possible local domino configurations in a tiling of G. In each case, as we traverse the boundary of the domino, we encounter three arrows in the direction of traversal, and three arrows in the direction opposite to it.

To understand height functions, we tile the complement of the Aztec diamond in R^+ with horizontal dominoes, and call this tiling T^+. Every boundary vertex of G thus lies on the boundary of at least one domino in T^+.

Definition 6.4. A *height function* H_T is a unique assignment of integers to the vertices of G given a tiling T such that

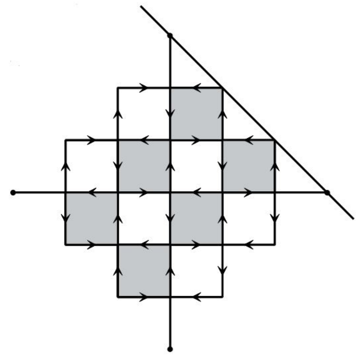


FIGURE 11. Colouring and orientation of G

(a) $H_T(-n-1, 0) = 0$
(b) if uv is the boundary of a domino in T^+ with $u \to v$, then $H_T(v) = H_T(u) + 1$

Height functions are characterised by the following two properties:

(a) Along the boundary cycles, H_T takes values $0, 1, \ldots, 2n+1, 2n+2, 2n+1, \ldots, 1, 0, 1, \ldots, 2n+1, 2n+2, 2n+1, \ldots, 1, 0$ starting from the vertex at $(-n-1, 0)$
(b) If $u \to v$, then $H_T(v)$ is either $H(u) + 1$ or $H(u) - 3$.

(a) is clear from the construction of T^+. To prove (b), observe that if uv is not a boundary of a tile in T^+, it must bisect a domino of T^+, in which case $H_T(v) = H_T(u) - 3$, which can be verified for each of the four possible domino configurations. If uv does lie on the boundary of T^+, then by the definition of H_T, $H_T(v) = H_T(u) + 1$. Conversely, if H is a height function that satisfies (a) and (b), then we can obtain the tiling T by placing a domino across every edge such that $|H(u) - H(v)| = 3$, and we conclude that $H = H_T$. Thus, we have a bijection between such height functions and domino tilings.

6.2. **Corner Sum and Height Function Matrices.**

Definition 6.5. A *corner sum matrix* of order n is an $(n+1) \times (n+1)$ matrix with entries in $\mathbb{N}$ such that

(a) the first row and column consist of 0s
(b) the last row and column consist of entries $0, \ldots, n$ in that order
(c) each entry is either equal to or one more than the entry to its left, and the entry above it

Proposition 6.6. *Corner sum matrices of order n are in bijection with ASMs of order n*

Proof. Let C_n be a corner sum matrix of order n. Then,

$$c_{i,j} = \sum_{\substack{0 \leqslant i' \leqslant i \\ 0 \leqslant j' \leqslant j}} a_{i',j'}$$

for some ASM $A_n = [a_{ij}]$ of order n. The inverse map $A_n \mapsto C_n$ is as follows:

$$a_{i',j'} = c_{i,j} - c_{i-1,j} - c_{i,j-1} + c_{i-1,j-1}$$

□

Definition 6.7. A *height function matrix* of order n is an $(n+1) \times (n+1)$ matrix with entries in $\mathbb{N}$ such that

(a) the first row and column consist of entries $0, \ldots, n$ in that order
(b) the last row and column consist of entries $n, \ldots, 0$ in that order
(c) every entry differs from the entry above and, and the entry to its left, by ± 1

Proposition 6.8. *Height function matrices of order n are in bijection with ASMs of order n*

Proof. Let $c_{i,j}$ be a corner sum matrix of order n. We can obtain the height function matrix $[h_{i,j}]$ by

$$h_{i,j} = i + j - 2c_{i,j}$$

Clearly, the inverse map $[h_{i,j}] \mapsto [c_{i,j}]$ exists. Thus, height function matrices of order n are in bjiection with corner sum matrices of the same order, which are further in bijection with ASMs of order n (proposition 6.6). The map $A_n \mapsto H_n$ is defined by the operation

$$a_{i',j'} = h_{i-1,j} + h_{i,j-1} - h_{i,j} - h_{i-1,j-1}$$

□

6.3. **Aztec diamonds and ASMs.** We will now show that domino tilings of the Aztec diamond region of order n (AD_n) are in bijection with pairs of ASMs (A, B), where A and B are of orders n and $n+1$ respectively, and satisfy a certain compatibility condition.

$(T \mapsto (A, B))$

Given a domino tiling T of AD_n with height function H_T, we construct (A, B) as follows:

- Define matrices A' and B' to record the values of H_T for odd and even vertices respectively:

$$\begin{aligned} A' = [a'_{i,j}] &= [H_T(-n+i+1, -i+j)] \\ B' = [b'_{i,j}] &= [H_T(-n-1+i+j, -i+j)] \qquad\qquad 0 \leqslant i, j \leqslant n \end{aligned}$$

  Since the boundary vertices have fixed heights, the first and last rows and columns of A' and B' are forced. Also note that consecutive entries in A' and B' must differ by exactly 2, as per the definition of H_T.
- From A' and B', construct matrices A^* and B^* as follows:

$$\begin{aligned} A^* = [a^*_{i,j}] &= \left[\frac{a'_{i,j} - 1}{2}\right] \\ B^* = [b^*_{i,j}] &= \left[\frac{b'_{i,j}}{2}\right] \end{aligned}$$

  A^* and B^* are height function matrices of orders n and $n+1$ respectively, and are thus in bijection with pairs of ASMs of the same orders.
- We can obtain the desired pair of ASMs, (A, B), from (A^*, B^*) as follows (proposition 6.2):

$$\begin{aligned} A = [a_{i,j}] &= [\frac{1}{2}(a^*_{i,j-1} + a^*_{i-1,j} - a^*_{i,j} - a^*_{i-1,j-1})] \\ B = [b_{i,j}] &= [\frac{1}{2}(b^*_{i,j-1} + b^*_{i-1,j} - b^*_{i,j} - b^*_{i-1,j-1})] \end{aligned}$$

Now, we define the inverse map. $((A, B) \mapsto T)$

Given ASMs A and B of orders n and $n+1$ respectively, we can reverse our steps and construct A' and B', and hence H_T.

The first and last rows and columns of A' and B' are forced, and record the heights of the boundary vertices of AD_n, which are fixed for any tiling T.

Consider an internal entry $b'_{i,j}$. A vertex with height $B_{i,j}$ is surrounded by vertices of heights $a'_{i,j}, a'_{i,j-1}, a'_{i-1,j}, a'_{i-1,j-1}$, and thus can differ from each of these by 1 or 3. It turns out that there are only 6 distinct configurations of these vertices:

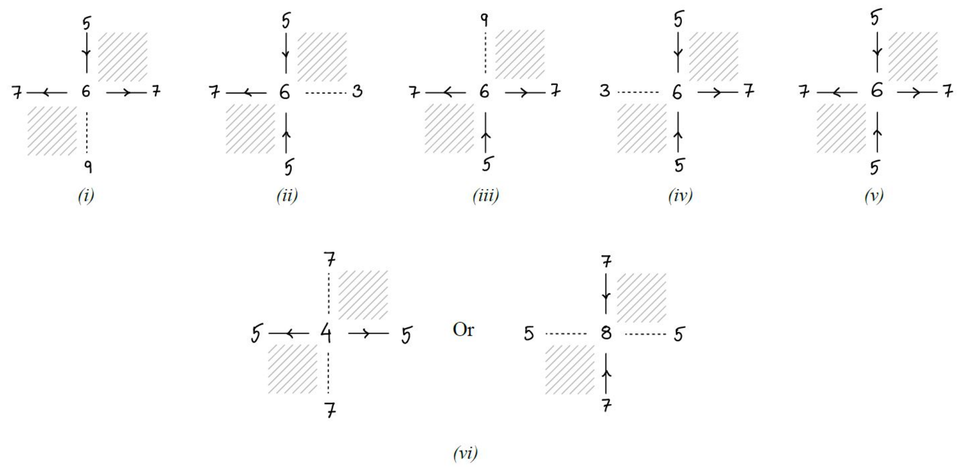


FIGURE 12. 6 local vertex configurations for an internal entry in $B_{i,j}$

Notice that $b'_{i,j}$ - and hence $b_{i,j}$ - is forced in all but one of these configurations. The case in which $b_{i,j}$ is not forced, corresponds to there being a (+1) in the $(i', j')^{th}$ entry of the ASM A. We can perform a similar case analysis for internal entries in A', and find that $a_{i,j}$ is not forced in only one of the 6 possible local configurations. This configuration corresponds to there being a (-1) in the $(i', j')^{th}$ entry of the ASM B.

Hence, if A is fixed and $A(\pm) =$ no. of $\pm 1s$ in A), we get $2^{n_+(A)}$ compatible ASMs B. Thus,

$$AD_n = \sum_{A \in ASM(n)} 2^{n_+(A)}$$

Equivalently, if B is fixed,

$$AD_n = \sum_{B \in ASM(n+1)} 2^{n_-(B)}$$

Replacing n by $n-1$, we get

$$AD_{n-1} = \sum_{B \in ASM(n)} 2^{n_-(B)}$$

In any ASM B, since each row sums to 1, $B(+) - B(-) = n$. Hence,

$$\begin{aligned} AD_n &= \sum_{A \in ASM(n)} 2^{n+A(-)} \\ &= 2^n \sum_{A \in ASM(n)} 2^{A(-)} \\ &= 2^n AD_{n-1} \end{aligned}$$

Therefore, by induction,

$$AD_n = 2^{\frac{n(n+1)}{2}}$$

References

[1] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, *"Alternating-Sign Matrices and Domino Tilings (Part I),"* Journal of Algebraic Combinatorics, vol. 1, no. 2, pp. 111–132, Sep. 1992

[2] D. Bressoud and J. Propp, *"How the Alternating Sign Matrix Conjecture Was Solved,"* vol. 46, no. 6, p. 10, 1999.

[3] D. M. Bressoud, *Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture*. Cambridge University Press, 1999.

[4] A. Arvind. *"MA 319 Lectures on Alternating Sign Matrices, Spring 2021."* YouTube, 2021, (https://www.youtube.com/playlist?list=PLIBms1v1GXKBfL9VaqHllmgovYE77lK18)

Dear Darius Singh,

It is my pleasure to extend the following offer of INTERNSHIP to you on behalf of **Rainet Technology Private Limited**, further to the interview and discussions you have had with us.

We hope that you find this offer acceptable. You are expected to join duty on 20 sep, 2021 and time duration for this internship for you is 3 month.

18th – September – 2021

To,

Mr. Singh,

Sub: **Letter of Internship**

Dear **Darius Singh,**

We are pleased to offer you an Internship with **The Tann Mann Foundation**. Your acceptance of this offer will make this a mutually binding contract.

Designation: **Deep Learning Engineer (Intern)**

Date of Joining: **19th September 2021**

Duration: 4 months

Location: **Work from home**

Remuneration: Volunteering for the charitable cause.

Non-Disclosure Clause: During this internship you are bound by Non-Disclosure clause meaning that you are not allowed to share any sensitive information of the organization with anyone without prior permission of the Management. Any breach can lead to termination and legal action.

We welcome you and wish you every success in your career with us.

Rahul

....	....
Mr. Rahul Nathan	**Mr. Darius Singh**
Trustee	Intern

The Tann Mann Foundation | www.thetannmanngaadi.org
Registered Office: M808 Brindavan Palms, Hosa Road, Bangalore - 560100
Mobile: +91 9341921581 I Email: WeCare@TheTannMannGaadi.Org

mycaptain

Certificate of Competency

This is to certify that

Niharika Garg

has been awarded Competency in MyCaptain's **Creative Writing Course**

Niharika Garg was guided by Captain Ujjwal Srivastava and they undertook the following projects in the same:

- Publish your first blog post
- Write a review for a book, movie, etc
- Create a micro fiction tale
- Write a monologue

For the month of December 2021

We wish you all the best for your future endeavours

Sameer Ramesh
(Chief Product Officer)

Certificate ID:
18N56RUJDHWPM

Course ratified by

KOMMUNE

An IIMB तेजस्वि नावधीतमस्तु NSRCEL incubated company A Climber Knowledge and Careers Pvt Ltd Initiative

mycaptain

3rd Floor, Classic Arena, Hosur Rd, AECS
Layout - A Block, Singasandra,
Bengaluru, Karnataka 560068

Contact :
support@mycaptain.in
1800 121 676767

anush.r@mycaptain.in
+91 8106104024

To: Whomsoever It May Concern

Subject: Letter of Recommendation for Niharika Garg

MyCaptain, an initiative by an NSRCEL, IIM Bangalore incubated company, is an online Learning platform awarded by the SDSN as one of the Top 50 youth led solutions working on Quality Education & Decent work and economic growth.

It is with great pleasure that I recommend Niharika Garg, on the basis of the performance in the MyCaptain Creative Writing Workshop in the month of December, 2021. Niharika Garg performed extremely well during the workshop and showed excellent skills as seen in the Project submissions and participation in the regular activities. As the mentor, I have seen that in every Live video class session, Niharika Garg was very participative and always willing to learn and enhance skills in every aspect. Niharika Garg is always willing to put forth a unique approach for the Projects that are assigned, while also adhering to the guidelines and rules.

With this, I would take this opportunity to wish Niharika Garg, all luck for future endeavors and hope for a bright future ahead.

Kind Regards,

Ujjwal Srivastava

Mentor Creative Writing Workshop

THE CLIMBER

Internship Offer with VEGA

Date: - January 10th, 2021

Dear Nandini,

I am delighted & excited to welcome you to VEGA Studios as a **Content Writing Intern**. At VEGA Studios, we believe that our team is our biggest strength and we take pride in hiring ONLY the best and the brightest. We are confident that you would play a significant role in the overall success of the venture and wish you the most enjoyable, learning packed and truly *meaningful* internship experience with VEGA Studios.

Your appointment will be governed by the terms and conditions presented in the **Annexure A.**

We look forward to you joining us. Please do not hesitate to call us for any information you may need. Also, please sign the duplicate of this offer as your acceptance and forward the same to us.

Congratulations!

Gajan Roy

Internship Offer with VEGA

Annexure A

You shall be governed by the following terms and condition of service during your internship with VEGA STUDIOS, and those may be amended from time to time.

1. You are being hired as a **Content Writing Intern.** As a Content Writing Intern you would be responsible for writing scripts of the provided movies.

2. Your date of joining is 10th January 2022 and the duration of the internship would be 1 month. During this time you are expected to devote your time and efforts solely to VEGA Studios work. You are also required to let your mentor know about forthcoming events (if there are any) in advance so that your work can be planned accordingly.

3. You will be working remotely for the duration of the internship. There will be catch ups scheduled with your mentor to discuss work progress and overall internship experience at regular intervals.

4. All the work that you will produce at or in relation to VEGA STUDIOS will be the intellectual property of VEGA STUDIOS. You are not allowed to store, copy, sell, share, and distribute it to a third party under any circumstances. Similarly you are expected to refrain from talking about your work in public domains (both online such as blogging, social networking site and offline among your friends, college etc.) without prior discussion and approval with your mentor.

5. We take data privacy and security very seriously and to maintain confidentiality of any students, customers, clients, and companies' data and contact details that you may get access to during your internship will be your responsibility. VEGA STUDIOS operates on **zero tolerance** principle with regard to any breach of data security guidelines. At the completion of the internship you are expected to hand over all VEGA STUDIOS work/data stored on your Personal Computer to your mentor and delete the same from your machine.

Internship Offer with VEGA

6. During the appointment period you shall not engage yourselves directly or indirectly or in any capacity in any other organization (other than your college). In the event of breach of this condition, this appointment is liable to be terminated forthwith by the company. In addition, you shall be liable to pay liquidated damages to the Company of an extent estimated by the Company.

7. Under normal circumstances either the company or you may terminate this association by providing a notice of 10 days without assigning any reason. However, the company may terminate this agreement forthwith under situations of in-disciplinary behaviours.

8. You are expected to conduct yourself with utmost professionalism in dealing with your mentor, team members, colleagues, clients and customers and treat everyone with due respect.

9. VEGA STUDIOS is a start-up and we love people who like to go beyond the normal call of the duty and can think out of the box. Surprise us with your passion, intelligence, creativity and hard work – and expect appreciation & rewards to follow.

10. Expect constant and continuous objective feedback from your mentor and other team members and we encourage you to ask for and provide feedback at every possible opportunity. It's your right to receive and give feedback – this is the ONLY way we all can continuously push ourselves to do better.

11. Have fun at what you do and do the right thing – both the principles are core of what VEGA STUDIOS stands for and we expect you to imbibe them in your day to day actions and continuously challenge us if we are falling short of expectations on either of them.

12. You will be provided **Rs 1000** per month as stipend and **Rs 100 – Rs 150** as an additional incentive based on the performance of the intern

13. Certificate for internship will be provided in a digital format once He/she completes the duration of internship

KYLO APPS

A Unit of Arihant Reclamation Pvt. Ltd.

KYLO APPS

Date: 08/09/2021

TO WHOMSOEVER IT MAY CONCERN

This is to certify that **Yashaswi Kafola** worked with KYLO APPS in the capacity of a Business Development Intern from 8 June 2021 to 8 September 2021.

He is a reliable and dedicated individual who would perform all the tasks with utmost diligence and deliver effective solutions within deadlines. Because of these qualities, he was also promoted to the position of Team Lead and to the position of Head of Business Development thereafter.

Yashaswi is an enthusiastic individual, who never ceases to bring out of the box ideas to the table and has excellent communication skills and a good rapport with his fellow interns and superiors. He has added value to our organization with his substantial contributions.

I recommend his good work and wish him all the luck and success for his future endeavours.

Saumya Thakur

Saumya Thakur
Founder - Kylo Apps

Contact- +91 8920607430 | +91 8860732624
hr@kyloapps.com
www.kyloapps.com

muskurahat foundation

CERTIFICATE OF COMPLETION

Presented To

Yashaswi Kafola

for recognition of your performance in Fundraising Internship of about One Month from **2nd June 2021 to 6th July 2021**, for the children catered by Muskurahat Foundation.

AMOUNT RAISED: Rs 1,100/-

HIMANSHU GOENKA
President & Founder

8th July 2021

Date